Distances between Formal Theories

Joint work with
Mohamed Khaled (Bahçeşehir University) and
Michelle Friend (George Washington University)

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Koen Lefever[1] Gergely Székely[2]

[1]Vrije Universiteit Brussel

[2]Alfréd Rényi Institute for Mathematics

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- Traditionally, *empirical adequacy* is the criterion to compare competing theories.
- However, this is not always straightforward:
  - Early Copernican heliocentrism provided less accurate predictions than the old Prolemaic geocentric model; see [Kuhn 1957].
  - One theory may be better suited to explain one part of the empirical data, another theory may be better suited to explain other parts of reality.
- We will here present another way to approach the comparison of theories, independent of empirical adequacy.

A motivating example

$$\text{Kin} := \{\text{AxEField}, \text{AxEv}, \text{AxSelf}, \text{AxSymD}, \text{AxLine}, \text{AxTriv}, \text{AxNoAcc}\}$$

$$\text{ClassicalKin} := \text{Kin} \cup \{\text{AxEther}, \text{AbsTime}, \text{AxThExp}_+\}$$

$$\text{SpecRel} := \text{Kin} \cup \{\text{AxPh}_c, \text{AxThExp}\}$$

Theorem:

ClassicalKin $\vdash$ Worldview transformations are Galilean transformations.

Theorem: (Andréka–Madarász–Németi, 1998)

SpecRel $\vdash$ Worldview transformations are Poincaré transformations.

Definition

A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

Definition

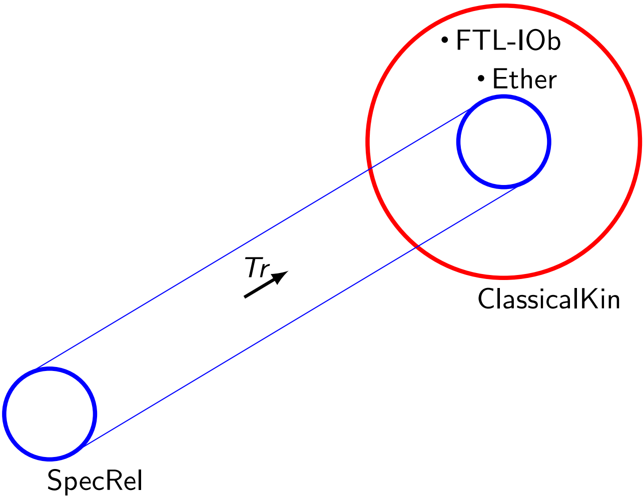
An *interpretation* of theory T_1 in theory T_2 is a translation Tr which translates all tautologies and all axioms of T_1 into theorems of T_2.

Definition

A *definitional equivalence* exists between two theories if those theories can be interpreted in each other and if all formulas from both theories translated into the other theory and back are logical equivalent to the original formulas.

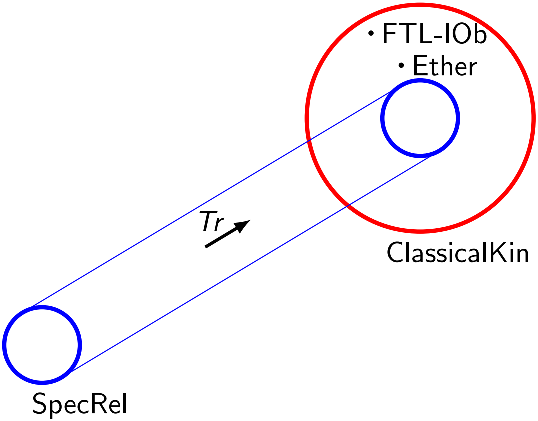
Theorem:

There is an interpretation *Tr* of SpecRel in ClassicalKin.



Theorem:

There is no definitional equivalence between SpecRel and ClassicalKin.



How can we make SpecRel and ClassicalKin equivalent?

Adding and removing concepts to make both theories equivalent:

- Removing FTL observers from ClassicalKin
- Adding a "primitive ether" to SpecRel

$$\mathsf{ClassicalKin}^{STL} := \mathsf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}^{\mathsf{STL}}, \mathsf{AxNoFTL}\}$$

AxNoFTL :

All inertial observers move slower than light with respect to the ether frames.

$$\neg\exists m\big(IOb(m) \land \exists e[Ether(e) \land Speed_e^{CK}(m) \geq \mathfrak{c}_{\mathfrak{e}}]\big).$$

AxThExp$^{\mathsf{STL}}$:

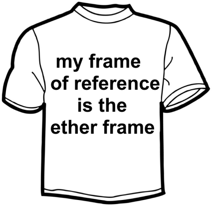
Inertial observers can move with any speed which is in the ether frame slower than that of light.

$$\exists h \ \big(IOb(h)\big) \land \forall e\bar{x}\bar{y}\big(Ether(e) \land \mathrm{space}(\bar{x},\bar{y}) < \mathfrak{c}_{\mathfrak{e}} \cdot \mathrm{time}(\bar{x},\bar{y}) \rightarrow \exists k \ \ IOb(k) \land W(e,k,\bar{x}) \land W(e,k,\bar{y})\big).$$

$$\mathsf{SpecRel}^e := \mathsf{SpecRel} \cup \{\mathsf{AxPrimitiveEther}\}$$

AxPrimitiveEther :

There is a non-empty class of ether observers, stationary with respect to each other, which is closed under trivial transformations.

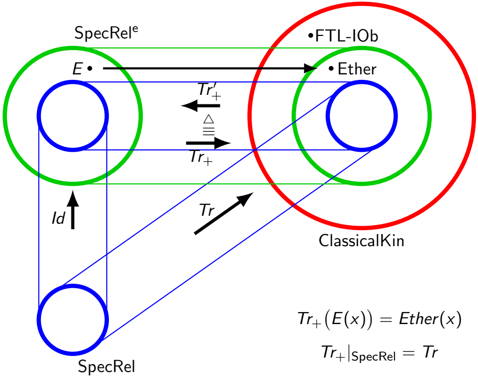


$$\exists e\big(E(e) \land \forall k\big[[IOb(k) \land (\exists T \in Triv) w_{ek}^{SR} = T] \leftrightarrow E(k)\big]\big)$$

Theorem:

There is an interpretation Tr'_+ of ClassicalKin^{STL} in SpecRel^e.

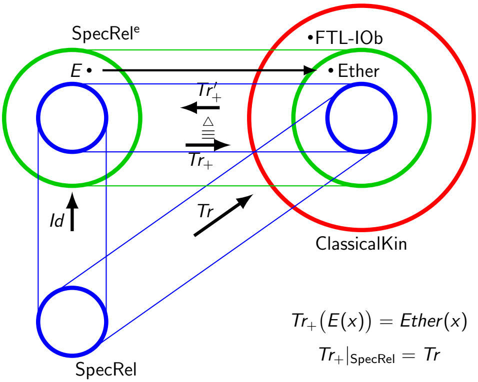
$$SpecRel^e \vdash Tr'_+(ClassicalKin^{STL})$$



Theorem:

Tr_+ and Tr'_+ are a definitional equivalence between SpecRel^e and ClassicalKin^{STL}.

$Tr'_+ \circ Tr_+(\text{SpecRel}^e) \Leftrightarrow \text{SpecRel}^e$ and $Tr_+ \circ Tr'_+(\text{ClassicalKin}^{STL}) \Leftrightarrow \text{ClassicalKin}^{STL}$



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\text{SpecRel}} = Tr$$

Theorem:

There is an interpretation Tr_* of ClassicalKin^{STL} in ClassicalKin.

$$\mathit{ClassicalKin} \vdash Tr_*(\mathit{ClassicalKin}^{STL})$$

Theorem:

There is an interpretation Tr'_* of ClassicalKin in ClassicalKin^{STL}.

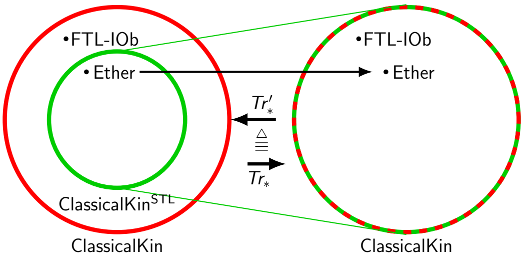
$$\mathit{ClassicalKin}^{STL} \vdash Tr'_*(\mathit{ClassicalKin})$$

Theorem:

Tr_* and Tr'_* are a definitional equivalence between ClassicalKin^{STL} and ClassicalKin.

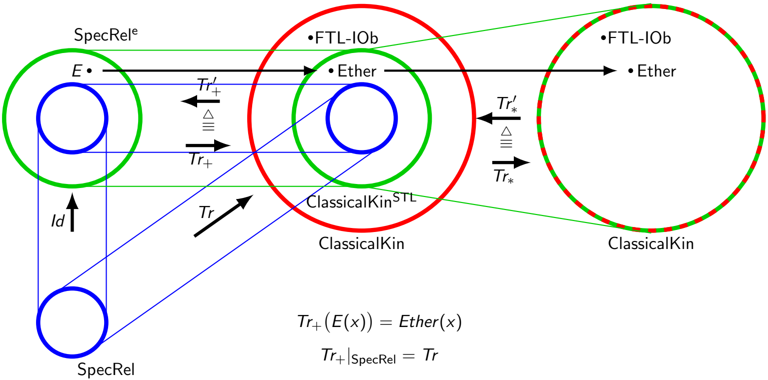
$Tr'_* \circ Tr_*(\text{ClassicalKin}^{STL}) \Leftrightarrow \text{ClassicalKin}^{STL}$ and

$Tr_* \circ Tr'_*(\text{ClassicalKin}) \Leftrightarrow \text{ClassicalKin}$



Theorem:

$Tr_* \circ Tr_+$ and $Tr'_+ \circ Tr'_*$ are a definitional equivalence between SpecRel^e and ClassicalKin.



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\mathrm{SpecRel}} = Tr$$

Equivalence relations:

There are several ways in which theories can be equivalent to each other:

Logical Equivalence ($\equiv$)

$\Downarrow\not\Uparrow$

Definitional Equivalence ($\stackrel{\triangle}{\equiv}$)

$\Downarrow\not\Uparrow$

Conceptual Equivalence (Morita Equivalence, definitional equivalence with different sorts)

$\Downarrow\not\Uparrow$

Categorical Equivalence

See [Barrett & Halvorson 2016].

Equivalence relations gives us a trivial (discrete) distance:

Let X-be any set of theories and E any equivalence relation on X. The *discrete distance* on (X, E) is the following:

$$d(x,y) = \begin{cases} 0 & \text{if } E(x,y), \\ 1 & \text{if } \neg E(x,y). \end{cases}$$

We are going to refine these trivial distances given by equivalence relations between theories.

We define *distances* between two theories based on the minimum number of things (e.g., axioms or concepts) which need to be added to or substracted from one theory to make it (logically, definitionally, etc.) equivalent to the other theory.

Two theories which are equivalent have a distance of zero.

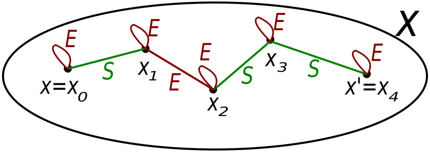
Definition

A **cluster network** is a triple (X, E, S), where X is a class with an equivalence relation E and a symmetric relation S.

$E(x, x')$: x and x' are basically the same objects.

$S(x, x')$: x or x' can be reached from the other in one "step."

A **path** *between* x *and* x' *in cluster network* (X, E, S) is a finite sequence of joining E-edges and S-edges connecting x and x'.



The **length** *of a path* is the number of S-edges in the path.

Objects $x, x' \in X$ *are* **connected** *in* (X, E, S) iff there is a path from one of them to the other.

Let $\mathcal{X} = (X, E, S)$ be a cluster network.

Definition

The **step distance** *on* $\mathcal{X}$ is the function $d_{\mathcal{X}} : X \times X \to \mathbb{N} \cup \{\infty\}$ defined as:

$$d_{\mathcal{X}}(x, x') \stackrel{\text{def}}{\equiv} \min\{k \in \mathbb{N} : \exists \text{ a path from } x \text{ to } x' \text{ of length } k\}$$

if x and x' are connected in $\mathcal{X}$, and

$$d_{\mathcal{X}}(x, x') \stackrel{\text{def}}{\equiv} \infty \quad \text{otherwise}$$

for each $x, x' \in X$.

Theorem:

Let $d_{\mathcal{X}} : X \times X \to \mathbb{N} \cup \{\infty\}$ be the step distance on cluster network $\mathcal{X} = (X, E, S)$. Then for each $x, y, z \in X$,

(a) $d_{\mathcal{X}}(x, y) \geq 0$, and $d_{\mathcal{X}}(x, y) = 0 \iff x\,E\,y$.

(b) $d_{\mathcal{X}}(x, y) = d_{\mathcal{X}}(y, x)$.

(c) $d_{\mathcal{X}}(x, y) \leq d_{\mathcal{X}}(x, z) + d_{\mathcal{X}}(z, y)$.

Axiomatic distance

Adding one axiom:

$$T \leftharpoondown T' \text{ iff there is } \varphi \in \mathrm{Fm} \text{ such that } T \cup \{\varphi\} \equiv T'.$$

One axiom difference:

$$T \rightleftharpoons T' \text{ iff either } T \leftharpoondown T' \text{ or } T' \leftharpoondown T.$$

Definition

Let $\mathcal{T}$ be a class of theories. We call the step distance on this cluster network $(\mathcal{T}, \equiv, \rightleftharpoons)$ the **axiomatic distance** $\mathrm{Ad}_{\mathcal{T}}$ *on* $\mathcal{T}$.

Example

Let $\mathcal{T}$ be a class of theories. Let $T, T_\perp \in \mathcal{T}$ be two theories formulated in the same language. Suppose that T is consistent while $T_\perp$ is inconsistent, then $\mathrm{Ad}_{\mathcal{T}}(T, T_\perp) = 1$.

Consequently, if $T, T' \in \mathcal{T}$ are formulated on the same language and an inconsistent theory $T_\perp$ of that language is in $\mathcal{T}$, then $\mathrm{Ad}_{\mathcal{T}}(T, T') \leq 2$ since we have $T \rightharpoonup T_\perp \leftharpoondown T'$.

Let CON be the class of all consistent theories.

Proposition:

Let $T, T' \in \mathrm{CON}$ having the same language $\mathcal{L}$. Then the axiomatic distance $\mathrm{Ad}_{\mathrm{CON}}(T, T') \leq 3$.

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Moreover, $\mathrm{Ad}_{\mathrm{CON}}(T, T') = 2$ if $T \not\equiv T'$ and they are complete.

Corollary:

Let $T, T' \in \mathrm{CON}$. Then,

$$\mathrm{Ad}_{\mathrm{CON}}(T, T') = \infty$$

$$\Updownarrow$$

T and T' are formulated in different langauges.

Theorem:

Fix a language $\mathcal{L}$. Let $T \in \mathrm{CON}$ be any theory on language $\mathcal{L}$ and let $\emptyset_{\mathcal{L}}$ be the empty theory on $\mathcal{L}$. Then:

1. $\mathrm{Ad}_{\mathrm{CON}}(\emptyset_{\mathcal{L}}, T) = 0$ iff T is trivial ($T \equiv \emptyset_{\mathcal{L}}$).
2. $\mathrm{Ad}_{\mathrm{CON}}(\emptyset_{\mathcal{L}}, T) = 1$ iff T is finitely axiomatizable and non-trivial.
3. $\mathrm{Ad}_{\mathrm{CON}}(\emptyset_{\mathcal{L}}, T) = 2$ iff T is not finitely axiomatizable, but has a finitely axiomatizable consistent extension.
4. $\mathrm{Ad}_{\mathrm{CON}}(\emptyset_{\mathcal{L}}, T) = 3$ iff T has no finitely axiomatizable consistent extension.

Theorem:

Fix a language $\mathcal{L}$. Let $T \in \mathrm{CON}$ be any theory on language $\mathcal{L}$ and let $\emptyset_{\mathcal{L}}$ be the empty theory on $\mathcal{L}$. Then:

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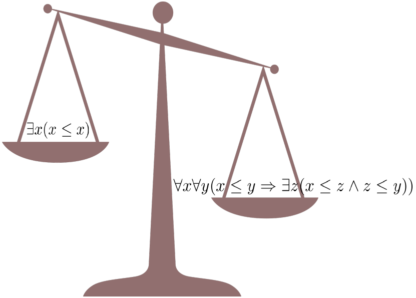
4. $\mathrm{Ad}_{\mathrm{CON}}(\emptyset_{\mathcal{L}}, T) = 3$ iff T has no finitely axiomatizable consistent extension.

Example

$\mathrm{Ad}_{\mathrm{CON}}(\emptyset_{\mathcal{L}}, \mathrm{PA}) = 3$, because PA (Peano Arithmetic) has no finitely axiomatizable consistent extension, see [Ryll-Nardzewski 1952].



By giving weights to the axiom adding steps, e.g., considering the addition of certain kinds of axioms as two or more steps, one could increase the resolution of axiomatic distance.

Conceptual distance

Definition

Theory T' of language $\mathcal{L}'$ is a **one-concept-extension** *of theory T of language $\mathcal{L}$:*

$$T \rightsquigarrow T' \stackrel{def}{\iff} \mathcal{L}' = \mathcal{L} \cup \{R\} \text{ and } (\forall \varphi \in \mathcal{L})(T' \models \varphi \text{ iff } T \models \varphi)$$

for some relation symbol R.

$$T \leftrightsquigarrow T' \stackrel{def}{\iff} T \rightsquigarrow T' \text{ or } T' \rightsquigarrow T$$

Definition

Let $\mathcal{T}$ be a class of theories. The **conceptual distance** $\mathrm{Cd}_{\mathcal{T}}$ is the step distance on cluster network $(\mathcal{T}, \stackrel{\triangle}{\equiv}, \leftrightsquigarrow)$, where $\stackrel{\triangle}{\equiv}$ is the definitional equivalence.

If $\mathcal{T}$ is the class of all theories of FOL, we omit the subscript $\mathcal{T}$ and simply write Cd.

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Theorem:

For every $n \in \mathbb{N} \cup \{\infty\}$, there are theories T and T' such that $\mathrm{Cd}(T, T') = n$.

Definition

The **spectrum** *of theory* T, in symbols $\mathrm{I}(T,\kappa)$, is the number of its different models of T (up to isomorphism) of cardinality κ.

Theorem:

If T_1 and T_2 are formulated in countable languages, then:

$$\mathrm{Cd}(T_1, T_2) < \infty$$

$$\Downarrow$$

$$(\forall \text{ cardinal } \kappa)\left[\mathrm{I}(T_1, \kappa) \neq 0 \iff \mathrm{I}(T_2, \kappa) \neq 0\right].$$

Theorem:

If T_1 and T_2 are formulated in finite languages, then:

$$\mathrm{Cd}(T_1, T_2) < \infty$$

$$\Updownarrow$$

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$$\Updownarrow$$

$$(\forall \text{ cardinal } \kappa) \left[\mathrm{I}(T_1, \kappa) \neq 0 \iff \mathrm{I}(T_2, \kappa) \neq 0\right].$$

Corollary:

There are infinitely many theories that are, in terms of conceptual distance, infinitely far from each other.

Proof.: If $\mathfrak{A}$ and $\mathfrak{B}$ are two finite models of different cardinality, then $\mathrm{Cd}\big(\mathrm{Th}(\mathfrak{A}), \mathrm{Th}(\mathfrak{B})\big) = \infty$. Q.E.D.

Theorem:

If T_1 and T_2 are formulated in finite languages, then

$$\left[\mathsf{Cd}(T_1, T_2) < \infty \text{ and } \mathsf{I}(T_1, 1) = \mathsf{I}(T_2, 1) = 0\right]$$

$$\Downarrow$$

$$\mathsf{Cd}(T_1, T_2) \leq 2.$$

Theorem:

$$\mathrm{Cd}(\mathsf{ClassicalKin}, \mathsf{SpecRel}) = 1$$

Proof.:

$$\mathit{SpecRel} \rightsquigarrow \mathit{SpecRel}^e \triangleq \mathit{ClassicalKin}^{STL} \triangleq \mathit{ClassicalKin}$$

Q.E.D.

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