Inelastic Collisions

Beginning a comparison of classical and relativistic dynamics in First-Order-Logic...

...and being sidetracked into variable independence of concepts

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Overview

- Previous result on comparing classical and relativistic kinematics
- Work-in-progress on dynamics
- Being side-tracked on variable-independent concepts

Definition

A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

Definition

An *interpretation* of theory T_1 in theory T_2 is a translation Tr which translates all tautologies and all axioms of T_1 into theorems of T_2.

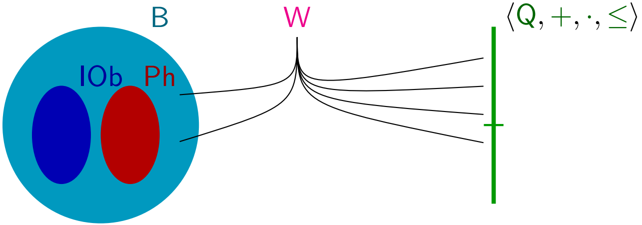
Definition

A *definitional equivalence* exists between two theories if those theories can be interpreted in each other and if all formulas from both theories translated into the other theory and back are logical equivalent to the original formulas.

Previous results on classical and relativistic kinematics

- K. Lefever: "*Using Logical Interpretation and Definitional Equivalence to Compare Classical Kinematics and Special Relativity Theory*"
  PhD Dissertation, Vrije Universiteit Brussel (2017).
- K. Lefever and G. Székely: "*Comparing Classical and Relativistic Kinematics in First-Order Logic*", Logique et Analyse, ISSN: 2295-5836, p. 57-117, Vol 61, Nr 241 (2018).

Language: $\{\, \mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \mathsf{W} \,\}$

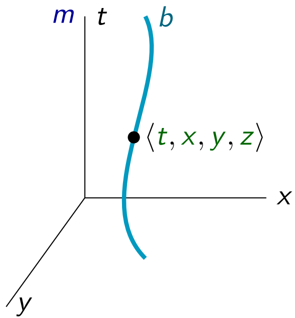


B ⇜⇝ Bodies (things that move)
IOb ⇜⇝ Inertial Observers   Ph ⇜⇝ Photons (light signals)

Q ⇜⇝ Quantities
$+$, $\cdot$ and $\leq$ ⇜⇝ field operations and ordering

W ⇜⇝ Worldview (a 6-ary relation of type BBQQQQ)

$W(m, b, t, x, y, z) \leftrightsquigarrow$ " observer m coordinatizes body b at spacetime location $\langle t, x, y, z\rangle$."



Worldline of body b according to observer m

$$wl_m(b) := \{\langle t, x, y, z\rangle \in Q^4 \ : \ W(m, b, t, x, y, z)\}$$

$$\mathsf{Kin} := \{\mathsf{AxEField}, \mathsf{AxEv}, \mathsf{AxSelf}, \mathsf{AxSymD}, \mathsf{AxLine}, \mathsf{AxTriv}, \mathsf{AxNoAcc}\}$$

$$\mathsf{ClassicalKin} := \mathsf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}_+\}$$

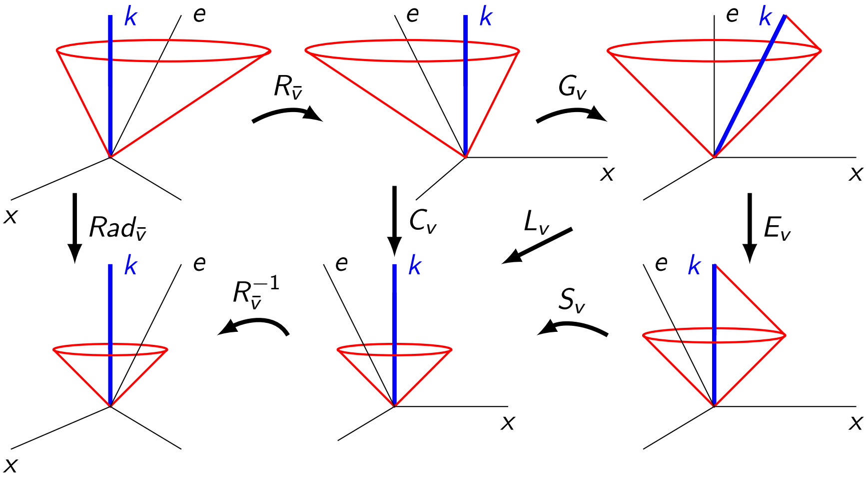
$$\mathsf{SpecRel} := \mathsf{Kin} \cup \{\mathsf{AxPh_c}, \mathsf{AxThExp}\}$$

Theorem:

ClassicalKin $\vdash$ Worldview transformations are Galilean transformations.

Theorem: (Andréka–Madarász–Németi, 1998)

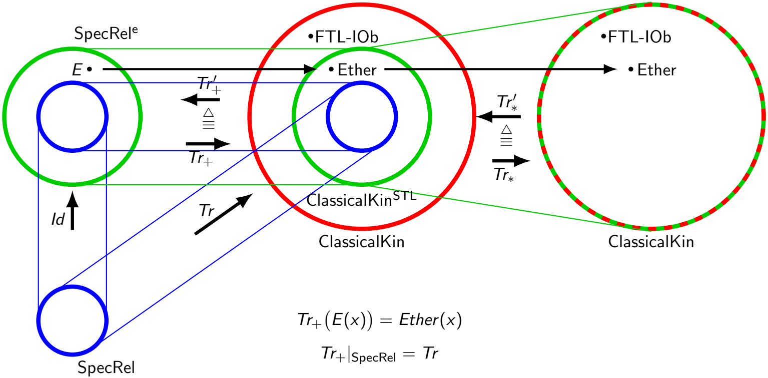
SpecRel $\vdash$ Worldview transformations are Poincaré transformations.



$$Rad_{\bar{v}} = R_{\bar{v}}^{-1} \circ S_v \circ E_v \circ R_{\bar{v}} \circ G_{\bar{v}}$$

Theorem:

SpecRel^e and ClassicalKin are definitionally equivalent.



$Tr_+(E(x)) = Ether(x)$

$Tr_+|_{\text{SpecRel}} = Tr$

Extending our results to dynamics

Inelastic collisions provide a simple case study, we need the following new concepts on top of those from kinematics:

- Mass: this also gives us impuls $\bar{P} = m\bar{v}$ and force $\bar{F} = m\bar{a}$
- Inertial bodies and inertial particles
- incoming and outgoing bodies at inelastic collisions
- New Axioms:
  - AxMass
  - AxSpeed
  - AxColl

Language: $\{\,\mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \mathsf{W}, \mathsf{M}\,\}$

Definition

M (the mass relation) is a 3-place relation symbol the first two arguments of which are of sort B and the third argument is of sort Q, reading $\mathsf{M}(k, b, q)$ as "the mass of body b is q according to observer k."

Definition

The relativistic mass of body b according to inertial observer k, in symbols $m_k(b)$, is defined to be q if $\mathsf{M}(k, b, q)$ holds and there is only one such $q \in \mathsf{Q}$; otherwise $m_k(b)$ is undefined.

Definition

If all inertial observers k which are stationary with respect to body b agree on its mass, then $m_0(b) = m_k(b)$ is the rest mass of body b.

From this we can define 4-velocity V and 4-momentum P:

Definition

$P_k(b) = \big(\mathfrak{c}_{\mathfrak{e}} m_k(b), m_k(b)\bar{v}_k(b)\big) = m_0(b)V_k(b).$

Body b is an inertial body if for all inertial obsevers k, the worldline $wl_k(b)$ is a subset of a straight line:

Definition

$$\mathsf{IB}(b) \stackrel{def}{\iff} (\forall k \in \mathsf{IOb})\left(\exists \bar{x}, \bar{y} \in Q^4\right)\left(\forall \bar{z} \in Q^4\right)\Big(W(k, b, \bar{z})$$

$$\to (\exists a \in Q)(\bar{z} = a\bar{x} + (1-a)\bar{y})\Big).$$

Body b is called inertial particle according to observer k, in symbols $IP_k(b)$, iff b is an inertial body and $m_k(b)$ is defined for inertial observer k:

Definition

$$IP_k(b) \stackrel{def}{\iff} \mathsf{IB}(b) \wedge \mathsf{IOb}(k) \wedge (\exists m \in Q)\Big(\mathsf{M}(k, b, m)$$

$$\wedge(\forall m' \in Q)\big(\mathsf{M}(k, b, m') \to m' = m\big)\Big).$$

Body b is incoming at coordinate point $\bar{x}$ according to inertial observer k, iff b is an inertial particle, $\bar{x}$ is on the world-line of b, and the time component of each coordinate point on the world-line of b different from $\bar{x}$ is *less* than the time component of $\bar{x}$:

Definition

$$in_k(b,\bar{x}) \stackrel{def}{\Longleftrightarrow}$$

$$IP_k(b) \wedge \mathsf{W}(k,b,\bar{x}) \wedge \forall \bar{y}\big(\mathsf{W}(k,b,\bar{y}) \rightarrow [\bar{y}=\bar{x} \vee y_1 < x_1]\big)$$

The definition for outgoing is similar:

Definition

$$out_k(b,\bar{x}) \stackrel{def}{\Longleftrightarrow}$$

$$IP_k(b) \wedge \mathsf{W}(k,b,\bar{x}) \wedge \forall \bar{y}\big(\mathsf{W}(k,b,\bar{y}) \rightarrow [\bar{y}=\bar{x} \vee y_1 > x_1]\big)$$

The collision of inertial particles $a_1 \ldots a_n$ at some point $\bar{x}$ according to observer k, creating inertial particles $b_1 \ldots b_m$ is:

Definition

$$coll_k(a_1 \ldots a_n : b_1 \ldots b_m)_{@\bar{x}} \stackrel{def}{\iff} \bigwedge_{i=1}^{n} in_k(a_i, \bar{x}) \wedge \bigwedge_{i=1}^{m} out_k(b_i, \bar{x}) \wedge$$

$$\sum_{i=1}^{n} m_k(a_i) = \sum_{i=1}^{m} m_k(b_i) \wedge \sum_{i=1}^{n} m_k(a_i) \cdot v_k(a_i) = \sum_{i=1}^{m} m_k(b_i) \cdot v_k(b_i).$$

When we do not care where the collision occurs, we can drop the $@\bar{x}$ subscript:

Definition

$$coll_k(a_1 \ldots a_n : b_1 \ldots b_n) \stackrel{def}{\iff}$$

$$(\exists \bar{x} \in Q^4)\ coll_k(a_1 \ldots a_n : b_1 \ldots b_m)_{@\bar{x}}$$

AxMass :

If the relativistic masses and velocities of two inertial particles coincide for an inertial observer, then their relativistic masses coincide for every inertial observer

$$IOb(k) \wedge IOb(h) \wedge Ip(a) \wedge Ip(b) \wedge m_k(a) = m_k(b) \wedge v_k(a) = v_k(b) \\ \rightarrow m_h(a) = m_h(b);$$

AxSpeed :

If an inertial particle is moving with the same slower than light speed according to two inertial observers, then the relativistic masses of the particle are the same for them

$$IOb(k) \wedge IOb(h) \wedge Ip(b) \wedge v_k(b) < \mathfrak{c} \rightarrow m_k(b) = m_h(b);$$

AxColl :

For every coordinate point $\bar{x} \in Q^n$ and for all positive quantities $m_1 \ldots m_n, m'_1 \ldots m'_l$ and 3-vectors $\bar{v}_1, \ldots, \bar{v}_n, \bar{v}'_1 \ldots \bar{v}'_l \in Q^3$ such that

$$\sum_{i=1}^{n} m_i = \sum_{j=1}^{l} m'_j \quad \text{and} \quad \sum_{i=1}^{n} m_i \cdot \bar{v}_i = \sum_{j=1}^{l} m'_j \cdot \bar{v}'_j,$$

there are inertial particle $b_1 \ldots b_n$ and $b'_1 \ldots b'_l$ such that $m_k(b_i) = m_i$, $\bar{v}_k(b_i) = \bar{v}_i$, $m_k(b'_j) = m'_j$, and $v_k(b'_j) = v'_j$ for all $i \leq n$ and $j \leq l$, and $coll_k(b_1 \ldots b_n : b'_1 \ldots b'_l)@\bar{x}$.

Let us define $m_k^*(b)$ from $P_k^*(b)$ as $m_k^*(b) \stackrel{def}{=} \frac{P_k^*(b)_0}{\mathfrak{c}_\mathfrak{e}}$ and define $P_k^*(b)$ as the image of $P_k(b)$ by $Rad_{v_k(e)}$.
Then by linearity of Rad_v, we have $P_k^*(a) + P_k^*(b) = P_k^*(c)$ if $P_k(a) + P_k(b) = P_k(c)$.
This allows us to tranlate mass as follows:

Definition

$$Tr\big(\mathsf{M}^{sr}(k,b,m)\big) \stackrel{def}{=} (\forall e \in Ether)(\exists m' \in Q) \left(\mathsf{M}^{ck}(k,b,m') \land m = \frac{Rad_{\bar{v}_k^{ck}(e)}\big(\mathfrak{c}_\mathfrak{e} m', m'\bar{v}_k^{ck}(b)\big)_0}{\mathfrak{c}_\mathfrak{e}}\right).$$

Variable-independent concepts

Assuming ClassicalKin, all ether observers are stationary with respect to each other, and hence they agree on the speed of light.

Definition

$$EOI_b^{k_1,\ldots,k_n}[\varphi] \stackrel{def}{\iff}$$

$$ClassicalKin \models (\forall k_1,\ldots,k_n \in IOb)(\forall e,e' \in Ether)$$

$$[\varphi(e/b) \iff \varphi(e'/b)]$$

where $\varphi(e/b)$ means that b gets replaced by e in all free occurences of b in formula φ.

The following rules can be used to show the ether independence of complex formulas:

1. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[\neg\varphi]$.
2. If $*$ is a logical connective, then from $EOI_b^{k_1,\ldots,k_n}[\varphi]$ and $EOI_b^{h_1,\ldots,h_m}[\psi]$ follows $EOI_b^{k_1,\ldots,k_n,h_1,\ldots,h_m}[\varphi * \psi]$.
3. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[(\exists x \in Q)(\varphi)]$ and $EOI_b^{k_1,\ldots,k_n}[(\exists h \in B)(\varphi)]$.
4. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[(\forall x \in Q)(\varphi)]$ and $EOI_b^{k_1,\ldots,k_n}[(\forall h \in B)(\varphi)]$.

AxSelf :

Every Inertial observer is stationary according to himself.

$$(\forall k \in IOb)(\forall \bar{y} \in Q^4)\left[W(k,k,\bar{y}) \leftrightarrow y_1 = y_2 = y_3 = 0\right]$$

$$Tr(\mathrm{AxSelf}_{\mathrm{SR}}) \equiv$$

$$\forall k \Big(IOb(k) \wedge (\forall e \in Ether)\big(speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big) \to (\forall \bar{y} \in Q^4)\,(\forall e \in Ether)\left[W\left(k,k,Rad^{-1}_{\bar{v}_k(e)}(\bar{y})\right) \leftrightarrow y_1 = y_2 = y_3 = 0\right]\Big)$$

$$\equiv$$

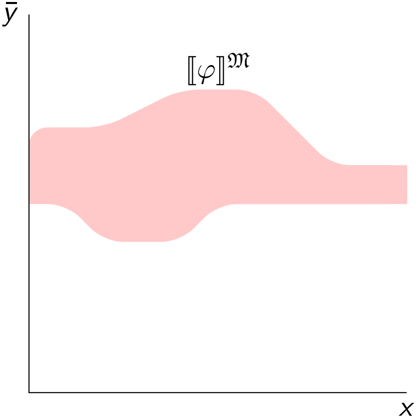
$$(\forall k \in IOb)(\forall e \in Ether)\Big(speed_e(k) < \mathfrak{c}_{\mathfrak{e}} \to (\forall \bar{y} \in Q^4)\left[W\left(k,k,Rad^{-1}_{\bar{v}_k(e)}(\bar{y})\right) \leftrightarrow y_1 = y_2 = y_3 = 0\right]\Big)$$

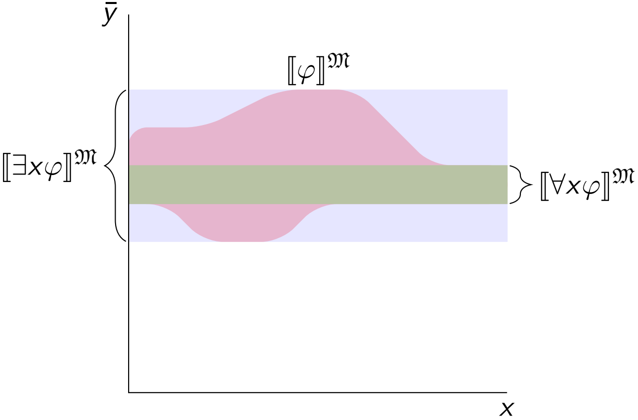
Let $\mathfrak{M}$ be a model and φ be a formula of its language. Then the *meaning of* φ *in* $\mathfrak{M}$ is defined as the set of sequences from $\mathfrak{M}$ satisfying φ, i.e.,

$$[\![\varphi]\!]^{\mathfrak{M}} \stackrel{\text{def}}{=} \{\bar{a} \in M^{\omega} : \mathfrak{M} \models \varphi[\bar{a}]\}.$$

In general,

$$[\![\forall x\varphi]\!]^{\mathfrak{M}} \subseteq [\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\exists x\varphi]\!]^{\mathfrak{M}}.$$


$\bar{y}$
$[\![\varphi]\!]^{\mathfrak{M}}$
x



$\bar{y}$
$[\![\varphi]\!]^{\mathfrak{M}}$
$[\![\exists x \varphi]\!]^{\mathfrak{M}}$
$[\![\forall x \varphi]\!]^{\mathfrak{M}}$
x

Let $\mathfrak{M}$ be a model, and let φ and θ be formulas in the language of $\mathfrak{M}$. We say that φ is independent of variable v_i *in* $\mathfrak{M}$ iff for all sequence of elements $\bar{a} \in M^\omega$ and and $b \in M$,

$$\mathfrak{M} \models \varphi[\bar{a}] \iff \mathfrak{M} \models \varphi[\bar{a}^i_b],$$

where $\bar{a}^i_b$ denotes the sequence which is the same as $\bar{a}$ except at i where it is b, i.e., $\bar{a}^i_b = (a_0, \ldots, a_{i-1}, b, a_{i+1}, \ldots)$.

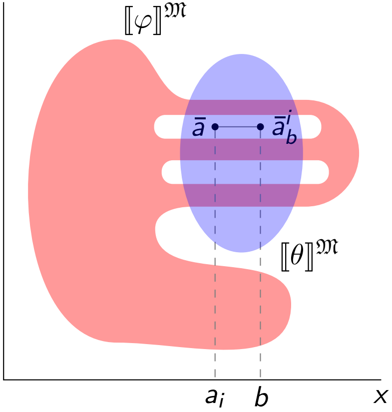
Proposition:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ be a formula of the language of $\mathfrak{M}$. Then the following statements are equivalent:

$$\varphi \text{ is independent of } x \text{ in } \mathfrak{M} \iff [\![\forall x\varphi]\!]^{\mathfrak{M}} = [\![\varphi]\!]^{\mathfrak{M}} \iff [\![\varphi]\!]^{\mathfrak{M}} = [\![\exists x\varphi]\!]^{\mathfrak{M}}.$$

We say that φ is independent of variable v_i *in* $\mathfrak{M}$ provided θ iff, for all sequence of elements $\bar{a} \in M^\omega$ and $b \in M$,

$$\mathfrak{M} \models \theta[\bar{a}] \text{ and } \mathfrak{M} \models \theta[\bar{a}^i_b] \implies (\mathfrak{M} \models \varphi[\bar{a}] \iff \mathfrak{M} \models \varphi[\bar{a}^i_b])$$



Another example: the Special Principle of Relativity

Scen – set of *experimental scenarios*.

Every experimental scenario $\varphi \in \mathrm{Scen}$ is either realizable by every *inertial observer* or by none of them.

$$\text{For all } \varphi \in \mathrm{Scen}: \quad IOb(k) \wedge IOb(k') \rightarrow \big[\varphi(k, \bar{x}) \leftrightarrow \varphi(k', \bar{x})\big].$$

- Judit X. Madarász, Gergely Székely and Mike Stannett: "Three Different Formalisations of Einstein's Relativity Principle", The Review of Symbolic Logic 10:(3) pp. 530-548. (2017)

Proposition:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ and θ be formulas of the language of $\mathfrak{M}$. Then the following statements are equivalent:

1. φ is independent of x in $\mathfrak{M}$ provided θ,
2. $[\![\theta \wedge \varphi]\!]^{\mathfrak{M}} = [\![\theta \wedge (\exists x \in \theta)\varphi]\!]^{\mathfrak{M}}$, and
3. $[\![\theta \to (\forall x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![\theta \to \varphi]\!]^{\mathfrak{M}}$.

Corollary:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ and ψ be formulas of the language of $\mathfrak{M}$. If ψ is independent of x in $\mathfrak{M}$, then

$$[\![\forall x(\varphi * \forall x\psi)]\!]^{\mathfrak{M}} = [\![\forall x(\varphi * \psi)]\!]^{\mathfrak{M}},$$

or in other words

$$\mathfrak{M} \models \forall x(\varphi * \forall x\psi) \leftrightarrow \forall x(\varphi * \psi).$$

Proposition:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ and θ be formulas of the language of $\mathfrak{M}$. Then the following statements are equivalent:

1. φ is independent of x in $\mathfrak{M}$ provided θ,
2. $[\![\theta \wedge \varphi]\!]^{\mathfrak{M}} = [\![\theta \wedge (\exists x \in \theta)\varphi]\!]^{\mathfrak{M}}$, and
3. $[\![\theta \to (\forall x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![\theta \to \varphi]\!]^{\mathfrak{M}}$.

Corollary:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ, ψ and θ be formulas of the language of $\mathfrak{M}$. If ψ is independent of x in $\mathfrak{M}$ provided θ, then

$$[\![\theta \wedge \varphi \to (\forall x \in \theta)\psi]\!]^{\mathfrak{M}} = [\![\theta \wedge \varphi \to \psi]\!]^{\mathfrak{M}}.$$

Corollary:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ, ψ and $\iota(\bar{y})$, $\epsilon(x)$ be formulas of the language of $\mathfrak{M}$. If ψ is independent of x in $\mathfrak{M}$ provided $\iota \wedge \epsilon$, then

$$[\![\iota \wedge \epsilon \wedge \varphi \to (\forall x \in \epsilon)\psi]\!]^{\mathfrak{M}} = [\![\iota \wedge \epsilon \wedge \varphi \to \psi]\!]^{\mathfrak{M}},$$

and hence

$$\mathfrak{M} \models (\forall \bar{y} \in \iota)(\forall x \in \epsilon)(\varphi \to (\forall x \in \epsilon)\psi)$$

$$\leftrightarrow (\forall \bar{y} \in \iota)(\forall x \in \epsilon)(\varphi \to \psi).$$

This is the generalized version of what we need to simplify our translated formulas: ι can represent the set of inertial observers, and ϵ can represent the set of ether frames.