On variable non-dependence of first-order formulas

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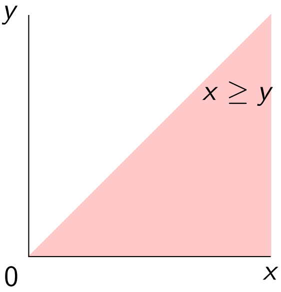
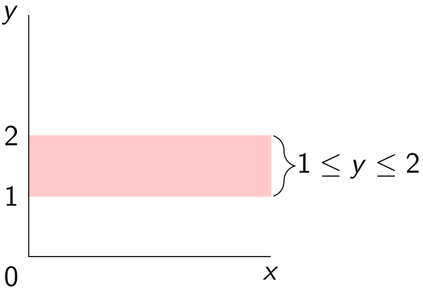
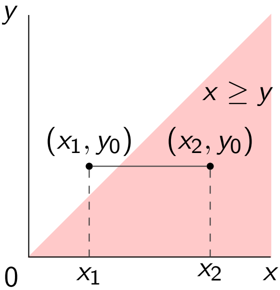
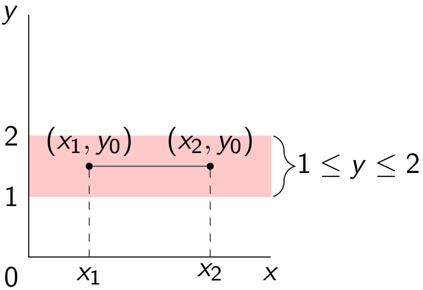
Concepts

Intuitively: a formula φ is *non-dependent of variable* x if the truth or falsity of formula φ does not depend on how variable x is interpreted, i.e., which value we assign to x.


y
$x \geq y$
0
x


y
2
1
$1 \leq y \leq 2$
0
x


y
x ≥ y
(x1, y0)
(x2, y0)
0
x1
x2
x


y
2
1
(x1, y0)
(x2, y0)
1 ≤ y ≤ 2
0
x1
x2
x

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- It is non-dependent of x provided some condition, e.g., $x(y^2-y) \geq 0$ is non-dependent of variable x in the ordered field of real numbers provided $x > 0$.

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To achieve non-dependence, we may need to put restrictions on the scope of interpretation of x and other variables. So in general, we say that φ is *non-dependent of variable x in a model provided some condition* captured by another formula θ.

Motivation

Ether-Observer-Independent formulas:

Assuming ClassicalKin, all ether observers are stationary with respect to each other, and hence they agree on the speed of light.

Definition

$$EOI_b^{k_1,\ldots,k_n}[\varphi] \overset{def}{\iff}$$

$$ClassicalKin \models (\forall k_1, \ldots, k_n \in IOb)(\forall e, e' \in Ether)$$

$$[\varphi(e/b) \iff \varphi(e'/b)]$$

where $\varphi(e/b)$ means that b gets replaced by e in all free occurences of b in formula φ.

The following rules can be used to show the ether independence of complex formulas:

1. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[\neg\varphi]$.
2. If $*$ is a logical connective, then from $EOI_b^{k_1,\ldots,k_n}[\varphi]$ and $EOI_b^{h_1,\ldots,h_m}[\psi]$ follows $EOI_b^{k_1,\ldots,k_n,h_1,\ldots,h_m}[\varphi * \psi]$.
3. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[(\exists x \in Q)(\varphi)]$ and $EOI_b^{k_1,\ldots,k_n}[(\exists h \in B)(\varphi)]$.
4. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[(\forall x \in Q)(\varphi)]$ and $EOI_b^{k_1,\ldots,k_n}[(\forall h \in B)(\varphi)]$.

AxSelf :

Every Inertial observer is stationary according to itself.

$$(\forall k \in \mathrm{IOb})(\forall t, x, y, z \in \mathrm{Q})\big[\mathrm{W}(k, k, t, x, y, z) \leftrightarrow x = y = z = 0\big].$$

$$Tr(\mathrm{AxSelf}_{\mathrm{SR}}) \equiv$$

$$(\forall k \in \mathrm{IOb})\underline{(\forall e \in Ether)}\Big(speed_e(k) < c$$
$$\rightarrow (\forall t, x, y, z \in \mathrm{Q})\underline{(\forall e \in Ether)}\big[\mathrm{W}\big(k, k, Rad^{-1}_{\bar{v}_k(e)}(k, k, t, x, y, z)\big) \leftrightarrow x = y = z = 0\big]\Big),$$

$$\equiv$$

$$(\forall k \in \mathrm{IOb})\underline{(\forall e \in Ether)}\Big(speed_e(k) < c$$
$$\rightarrow (\forall t, x, y, z \in \mathrm{Q})\big[\mathrm{W}\big(k, k, Rad^{-1}_{\bar{v}_k(e)}(k, k, t, x, y, z)\big) \leftrightarrow x = y = z = 0\big]\Big)$$

We want to generalize this kind of simplifications of complex mechanographically translated formulas to situations outside of special relativity.

Another example: the Special Principle of Relativity

A formulation of the principle in (Madarász et al. 2017, section 4.1) states that the truth of certain formulas $\varphi(b, \bar{x})$ describing experimental scenarios with numerical parameters $\bar{x}$ does not depend on the choice of inertial observer b:

$$\mathsf{IOb}(k) \wedge \mathsf{IOb}(h) \rightarrow (\varphi(k, \bar{x}) \leftrightarrow \varphi(h, \bar{x})).$$

Formal framework

Definition

For arbitrary formulas ϕ and φ, we are using bounded quantifiers as follows:

$$(\forall u \in \phi)\varphi \overset{def}{\iff} \forall u(\phi \rightarrow \varphi)$$

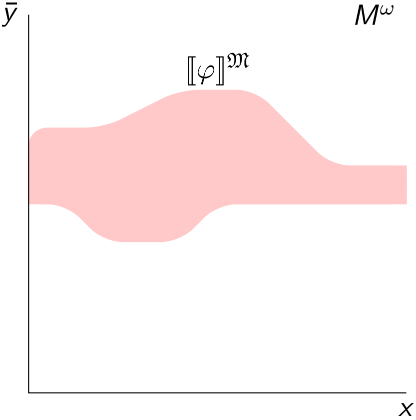
and

$$(\exists u \in \phi)\varphi \overset{def}{\iff} \exists u(\phi \cap \varphi).$$

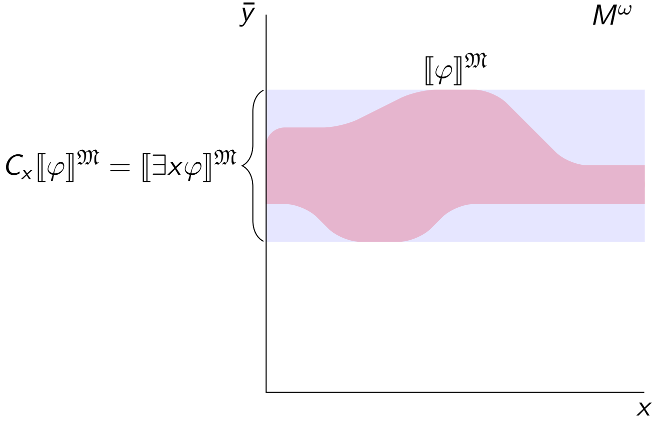
Definition

Let $\mathfrak{M}$ be a model and φ be a formula of its language. Then the **meaning** *of* φ *in* $\mathfrak{M}$ is defined as the set of sequences from $\mathfrak{M}$ satisfying φ, i.e.,

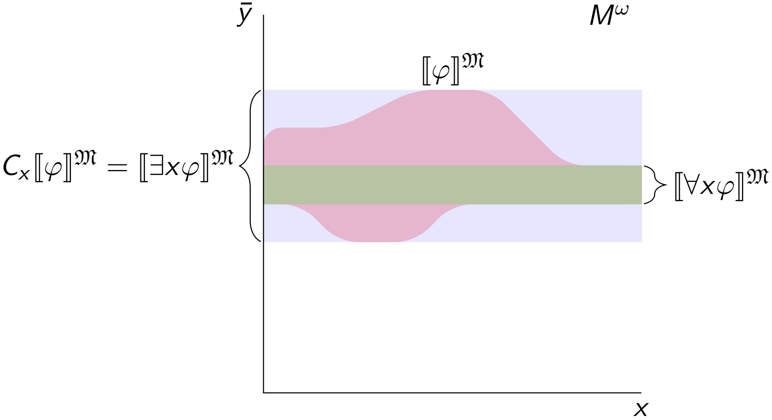
$$[\![\varphi]\!]^{\mathfrak{M}} \overset{\text{def}}{\equiv} \{\bar{a} \in M^{\omega} : \mathfrak{M} \models \varphi[\bar{a}]\}.$$



$$[\![\varphi]\!]^{\mathfrak{M}} \stackrel{\text{def}}{\equiv} \{\bar{a} \in M^\omega : \mathfrak{M} \models \varphi[\bar{a}]\}.$$



Cylindrification $C_x[\![\varphi]\!]^{\mathfrak{M}}$



$$[\![\forall x \varphi]\!]^{\mathfrak{M}} \subseteq [\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\exists x \varphi]\!]^{\mathfrak{M}}$$

Definition

By $\bar{a}^i_b$, let us denote the sequence which is the same as $\bar{a} = (a_1, a_2, \dots, a_n, \dots)$ except at i where it is b, i.e., $\bar{a}^i_b = (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots)$.

When using a metavariable, say x abbreviating v_i, we talk about the x-th component of $\bar{a}$ meaning the i-th component, and also use notation $\bar{a}^x_b$ instead of $\bar{a}^i_b$.

Definition

We will use the following notation for Tarski's substitution:

$$\varphi^x_y \stackrel{def}{\Longleftrightarrow} \exists x(x = y \wedge \varphi).$$

This definition of substitution is equivalent to Tarski's definition $\varphi(x/y) \stackrel{def}{\Longleftrightarrow} \forall x(x = y \rightarrow \varphi)$ in (Tarski, 1964 p.62), however we use Enderton's notation φ^x_y from (Enderton 2001, p. 112) in stead of Tarski's $\varphi(x/y)$.

Enderton's definition is equivalent with Tarski's for proper substitution, see (Enderton 2001, p.130).

Let us note that

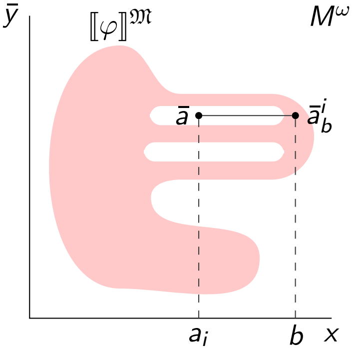
$$\mathfrak{M} \models \varphi^x_{v_j}[\bar{a}] \iff \mathfrak{M} \models \varphi[\bar{a}^x_{a_j}].$$

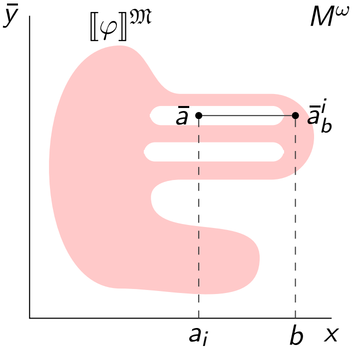
Definition of Variable Non-Dependence

Definition

φ is **non-dependent of variable** x *in model* $\mathfrak{M}$ iff for all sequences of elements $\bar{a} \in M^\omega$ and $b \in M$,

$$\mathfrak{M} \models \varphi[\bar{a}] \iff \mathfrak{M} \models \varphi[\bar{a}^x_b].$$


$\bar{y}$
$[\![\varphi]\!]^{\mathfrak{M}}$
M^{ω}
$\bar{a}$
$\bar{a}^i_b$
a_i
b
x


$\bar{y}$
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a_i
b
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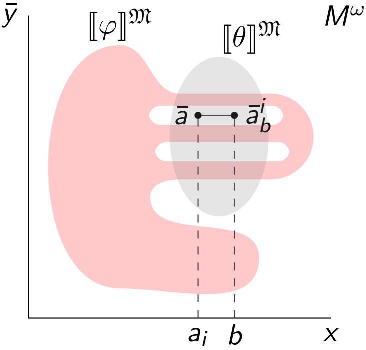

$\bar{y}$
$[\![\varphi]\!]^{\mathfrak{M}}$
$[\![\theta]\!]^{\mathfrak{M}}$
M^ω
$\bar{a}$
$\bar{a}^i_b$
a_i
b
x

Definition

φ is **non-dependent of variable** x **in model** $\mathfrak{M}$ **provided** θ iff, for all sequences of elements $\bar{a} \in M^\omega$ and $b \in M$,

$$\mathfrak{M} \models \theta[\bar{a}] \text{ and } \mathfrak{M} \models \theta[\bar{a}^x_b] \implies (\mathfrak{M} \models \varphi[\bar{a}] \iff \mathfrak{M} \models \varphi[\bar{a}^x_b]).$$

Results

Proposition:

The following statements are equivalent:

- φ is non-dependent of x in $\mathfrak{M}$ provided θ,
- $[\![\theta \wedge (\exists x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![\theta \wedge \varphi]\!]^{\mathfrak{M}}$,
- $[\![\theta \wedge (\forall x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![\theta \wedge \varphi]\!]^{\mathfrak{M}}$,
- $[\![\theta \rightarrow (\forall x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![\theta \rightarrow \varphi]\!]^{\mathfrak{M}}$, and
- $[\![\theta \rightarrow (\exists x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![\theta \rightarrow \varphi]\!]^{\mathfrak{M}}$.

Let us note here that

$$[\![\theta \wedge \exists x(\theta \wedge \varphi)]\!]^{\mathfrak{M}} \supseteq [\![\theta \wedge \varphi]\!]^{\mathfrak{M}} \subseteq [\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\theta \rightarrow \varphi]\!]^{\mathfrak{M}} \supseteq [\![\theta \rightarrow \forall x(\theta \rightarrow \varphi)]\!]^{\mathfrak{M}}$$

holds in general.

Corollary:

$$\begin{aligned}\varphi \text{ is non-dependent of } x \text{ in } \mathfrak{M} \\ \iff \llbracket \forall x \varphi \rrbracket^{\mathfrak{M}} = \llbracket \varphi \rrbracket^{\mathfrak{M}} \\ \iff \llbracket \varphi \rrbracket^{\mathfrak{M}} = \llbracket \exists x \varphi \rrbracket^{\mathfrak{M}}\end{aligned}$$

and hence

$$\varphi \text{ is non-dependent of } x \text{ in } \mathfrak{M} \iff \mathfrak{M} \models \exists x \varphi \leftrightarrow \forall x \varphi.$$

Let us note that if variable x does not occur free in φ, then φ is non-dependent of variable x in every model. However, the converse does not hold: for example, the formula $x = x$ is non-dependent of variable x in every model, but x does occur free in it.

Proposition:

If φ is non-dependent of x in $\mathfrak{M}$ provided θ, then $[\![(\exists x \in \theta)\neg\varphi)]\!]^{\mathfrak{M}}$ is the complement of $[\![(\exists x \in \theta)\varphi)]\!]^{\mathfrak{M}}$ relative to $[\![\exists x\theta]\!]^{\mathfrak{M}}$, i.e.,

$$[\![(\exists x \in \theta)\neg\varphi)]\!]^{\mathfrak{M}} = [\![\exists x\theta]\!]^{\mathfrak{M}} - [\![(\exists x \in \theta)\varphi)]\!]^{\mathfrak{M}}$$

in other words

$$[\![(\exists x \in \theta)\neg\varphi)]\!]^{\mathfrak{M}} = [\![\exists x\theta \wedge \neg(\exists x \in \theta)\varphi)]\!]^{\mathfrak{M}}.$$

Proposition:

If φ is non-dependent of x in $\mathfrak{M}$ provided θ, then

$$[\![(\forall x \in \theta)\neg\varphi)]\!]^{\mathfrak{M}} = [\![\exists x\theta \rightarrow \neg(\forall x \in \theta)\varphi)]\!]^{\mathfrak{M}}.$$

Corollary:

If φ is non-dependent of x in $\mathfrak{M}$ provided θ and $\mathfrak{M} \models \exists x\theta$, i.e., $[\![\exists x\theta]\!]^{\mathfrak{M}} = M^{\omega}$, then

$$[\![\neg(\exists x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![(\exists x \in \theta)\neg\varphi)]\!]^{\mathfrak{M}}, \text{ and}$$
$$[\![\neg(\forall x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![(\forall x \in \theta)\neg\varphi)]\!]^{\mathfrak{M}}.$$

Proposition:

If φ is non-dependent of x in $\mathfrak{M}$ provided θ and $\mathfrak{M} \models \exists x\theta$, then

$$[\![(\exists x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![(\forall x \in \theta)\varphi)]\!]^{\mathfrak{M}}.$$

Proposition:

Let f be any boolean expression, then

$$\llbracket \exists x\theta \to f\big((\forall x \in \theta)\varphi_1, \ldots, (\forall x \in \theta)\varphi_n\big) \rrbracket^{\mathfrak{M}} \\ = \llbracket (\forall x \in \theta) f(\varphi_1, \ldots, \varphi_n) \rrbracket^{\mathfrak{M}}$$

if all φ_i are non-dependent of x in $\mathfrak{M}$ provided θ.

Theorem:

Let f be any boolean expression, let θ be formula such that no variables of $z_1, \ldots, z_m$ occur free in θ, and let $Q_1, \ldots, Q_m$ be an arbitrary series of universal and existential quantifiers. Then, if all φ_i are non-dependent of x in $\mathfrak{M}$ provided θ,

$$\begin{aligned}&[\![\exists x\theta \rightarrow Q_m z_m \ldots Q_1 z_1 f\big((\forall x \in \theta)\varphi_1, \ldots, (\forall x \in \theta)\varphi_n\big)]\!]^{\mathfrak{M}}\\ &\qquad = [\![(\forall x \in \theta) Q_m z_m \ldots Q_1 z_1 f(\varphi_1, \ldots, \varphi_n)]\!]^{\mathfrak{M}},\end{aligned}$$

and hence, if $\mathfrak{M} \models \exists x\theta$, then

$$\begin{aligned}&[\![Q_m z_m \ldots Q_1 z_1 f\big((\forall x \in \theta)\varphi_1, \ldots, (\forall x \in \theta)\varphi_n\big)]\!]^{\mathfrak{M}} =\\ &\qquad [\![(\forall x \in \theta) Q_m z_m \ldots Q_1 z_1 f(\varphi_1, \ldots, \varphi_n)]\!]^{\mathfrak{M}}.\end{aligned}$$

Corollary:

If the conditions of the Theorem hold and $\mathfrak{M} \models \exists x\theta$, then also the existential quantifiers $(\exists x \in \theta)$ can be brought out from the corresponding formula, i.e.,

$$\llbracket Q_m z_m \dots Q_1 z_1 f\big((\exists x \in \theta)\varphi_1, \dots, (\exists x \in \theta)\varphi_n\big)\rrbracket^{\mathfrak{M}} = \llbracket(\exists x \in \theta) Q_m z_m \dots Q_1 z_1 f(\varphi_1, \dots, \varphi_n)\rrbracket^{\mathfrak{M}}.$$

Moreover, any of the quantifiers $(\exists x \in \theta)$ can freely be replaced by quantifiers $(\forall x \in \theta)$ in this equation above, which makes the application of the Theorem more general.

Application

AxSelf :

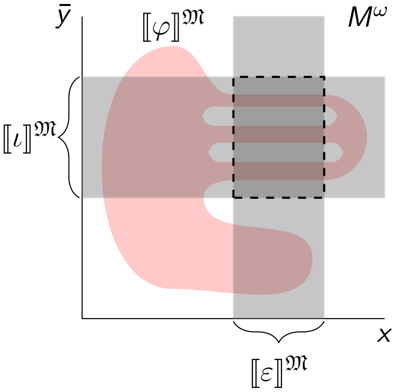
Every Inertial observer is stationary according to itself.

$$(\forall k \in \mathsf{IOb})(\forall t, x, y, z \in \mathsf{Q})\big[\mathsf{W}(k, k, t, x, y, z) \leftrightarrow x = y = z = 0\big].$$

The translated axiom $Tr(\mathsf{AxSelf}_{\mathsf{SR}}) \equiv$

$$(\forall k \in \mathsf{IOb})\underline{(\forall e \in Ether)}\Big(speed_e(k) < c$$
$$\rightarrow (\forall t, x, y, z \in \mathsf{Q})\underline{(\forall e \in Ether)}\big[\mathsf{W}\big(k, k, Rad^{-1}_{\bar{v}_k(e)}(k, k, t, x, y, z)\big) \leftrightarrow x = y = z = 0\big]\Big),$$

has the form $(\forall k \in \iota)\underline{(\forall e \in \varepsilon)}\big(\phi \rightarrow (\forall z \in \delta)\underline{(\forall e \in \varepsilon)}\psi\big)$.



The special case when the provided condition is of the form $\theta = \iota \wedge \varepsilon$ for some formulas such that x does not occur free in ι and certain bounded variables of φ do not occur free in ε and ι. Here $[\![\theta]\!]^{\mathfrak{M}} = [\![\iota]\!]^{\mathfrak{M}} \cap [\![\varepsilon]\!]^{\mathfrak{M}}$ is represented by the area inside the dashed rectangle.

Proposition:

Formula φ is non-dependent of x in $\mathfrak{M}$ provided θ iff

$$\mathfrak{M} \models (\theta^x_y \wedge \theta^x_z) \rightarrow (\varphi^x_y \leftrightarrow \varphi^x_z)$$

for some variable y and z that occur neither in φ nor in θ.

Theorem:

Let $\mathfrak{M}$ be a model, let f be any Boolean expression, let ι and ε be formulas such that variable x does not occur free in ι, and let $\varphi_1, \ldots, \varphi_n$ be formulas such that each of $\varphi_1, \ldots, \varphi_n$ is non-dependent of variable x in $\mathfrak{M}$ provided $\iota \wedge \varepsilon$ and $\mathfrak{M} \models \exists x \varepsilon$, and let $Q_1, \ldots, Q_k$ as well as $\bar{Q}$ be arbitrary series of universal and existential quantifiers, then

$$\llbracket (Q_1 u_1 \in \iota) \ldots (Q_k u_k \in \iota) \bar{Q} \bar{z} f\big((\forall x \in \varepsilon)(\varphi_1), \ldots, (\forall x \in \varepsilon)(\varphi_n)\big) \rrbracket^{\mathfrak{M}} = \llbracket (Q_1 u_1 \in \iota) \ldots (Q_k u_k \in \iota)(\forall x \in \varepsilon) \bar{Q} \bar{z} f\big(\varphi_1, \ldots, \varphi_n\big) \rrbracket^{\mathfrak{M}}$$

if no variables of $\bar{z}$ occur free in ι and ε.

Corollary:

We can simpilfy $(\forall k \in \iota)\underline{(\forall e \in \varepsilon)}\big(\phi \to (\forall z \in \delta)\underline{(\forall e \in \varepsilon)}\psi\big)$ to $(\forall k \in \iota)\underline{(\forall e \in \varepsilon)}\big(\phi \to (\forall z \in \delta)\psi\big)$ if ι and ε are formulas in the language of model $\mathfrak{M}$ such that variable e does not occur free in ι, if ϕ, δ and ψ are formulas which are non-dependent of variable e in $\mathfrak{M}$ provided $\iota \wedge \varepsilon$, and if $\mathfrak{M} \models \exists e \varepsilon$.

$$(\forall k \in \mathsf{IOb})\underline{(\forall e \in Ether)}\Big(speed_e(k) < c$$

$$\to (\forall t, x, y, z \in \mathsf{Q})\underline{(\forall e \in Ether)}\big[\mathsf{W}\big(k, k, Rad^{-1}_{\bar{v}_k(e)}(k, k, t, x, y, z)\big) \leftrightarrow x = y = z = 0\big]\Big),$$

$$\equiv$$

$$(\forall k \in \mathsf{IOb})\underline{(\forall e \in Ether)}\Big(speed_e(k) < c$$

$$\to (\forall t, x, y, z \in \mathsf{Q})\big[\mathsf{W}\big(k, k, Rad^{-1}_{\bar{v}_k(e)}(k, k, t, x, y, z)\big) \leftrightarrow x = y = z = 0\big]\Big)$$

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