VRIJE
UNIVERSITEIT
BRUSSEL

Using Logical Interpretation and Definitional Equivalence to Compare Classical Kinematics and Special Relativity Theory

PhD Dissertation

Koen Lefever

May 26, 2017

Promotor: Prof. Dr. Jean Paul Van Bendegem
Center for Logic and Philosophy of Science
Vrije Universiteit Brussel

Co-promotor: Dr. Gergely Székely
Alfréd Rényi Institute for Mathematics
Hungarian Academy of Sciences

Letteren & Wijsbegeerte

ACKNOWLEDGEMENTS

In the first place, I want to thank my promotors Prof. Dr. Jean Paul Van Bendegem and Dr. Gergely Székely, whithout whose extraordinary support and patience writing this dissertation would have been impossible.

I want to thank Prof. Dr. Maarten Van Dyck of Universiteit Gent for joining my promotors in the PhD advisory commission.

I am grateful to Dr. Marcoen Cabbolet, Prof. Dr. Steffen Ducheyne, Prof. Dr. Michele Friend, Prof. Dr. Sonja Smets and Prof. Dr. Bart Van Kerkhove for accepting to be in the jury and for helpful feedback.

I am grateful to Prof. Dr. Hajnal Andréka, Mr. Péter Fekete, Dr. Judit X. Madarász, Prof. Dr. István Németi and Prof. Dr. Laszló E. Szabó for enjoyable discussions and helpful suggestions while writing this dissertation.

In Budapest, I have enjoyed the hospitality of the Alfréd Rényi Institute for Mathematics as visiting scholar in the summers of 2014 and 2016, the unfortunately now defunct ELTE Peregrinus Hotel, and Mrs. Anett Bolvári while working on my research. Also, the participants of the Logic, Relativity and Beyond conferences in Budapest have been a great inspiration.

The encouragement of my family, in particular of my children Lilith & Zenobius and my parents Amedé & Christiane, has been great. The effort that my father did by studying relativity theory to support, show interest in and understand my work is much appreciated.

My colleagues, former colleagues, former teachers, former professors and former students at the Vrije Universiteit Brussel, the Faculty of Arts and Philosophy, the Center for Logic and Philosophy of Science, Vesalius College Brussels and Ghent University have also been most supportive.

There are too many friends and others who have been encouraging me to name them all. They know who they are, and they know I am grateful.

This dissertation has been produced using Free software running on GNU/Linux. To edit the LaTeX files, we have been using Texmaker & Emacs. The figures have been created using kTikZ. For version control, we have been using Git. We have been communicating using jitsi.meet & qTox.

Finaly, I want to thank everybody who has been teaching me about wisdom, strength and beauty.

ABSTRACT

This dissertation is a study in *philosophy of science* on using methods from mathematical logic to compare different, non-equivalent scientific theories, applied to two important theories in physics.

The aim is to present a new logic based understanding of the connection between special relativity and classical kinematics.

We show that the axioms of special relativity can be interpreted in the language of classical kinematics. This means that there is a logical translation function from the language of special relativity to the language of classical kinematics which translates the axioms of special relativity into consequences of classical kinematics. This is not in contradiction with physics, as some key concepts such as simultaneity, time difference or distance are changed by this translation.

We will also show that if we distinguish a class of observers which are stationary relative to each other (representing the observers stationary with respect to the "Ether") in special relativity and exclude the non-slower-than light observers from classical kinematics by an extra axiom, then the two theories become definitionally equivalent (i.e., they become equivalent theories in the sense as the theory of lattices as algebraic structures is the same as the theory of lattices as partially ordered sets).

Furthermore, we show that classical kinematics is definitionally equivalent to classical kinematics with only slower-than-light inertial observers, and hence by transitivity of definitional equivalence that special relativity theory extended with "Ether" is definitional equivalent to classical kinematics.

So within an axiomatic framework of mathematical logic, we explicitly show that the transition from classical kinematics to special relativity is the knowledge acquisition of that there is no "Ether".

★

"*La pensée ne doit jamais se soumettre, ni à un dogme, ni à un parti, ni à une passion, ni à un intérêt, ni à une idée préconue, ni à quoi que ce soit, si ce n'est aux faits eux-mêmes, parce que, pour elle, se soumettre, ce serait cesser d'être.*"

Henri Poincaré, speech given on 19 November 1909 at the 75$^{\text{th}}$ birthday of the Université Libre de Bruxelles, [37, p. 152].

CONTENTS

1. INTRODUCTION

The aim of this dissertation is to provide a new, deeper and more systematic understanding of the connection of classical kinematics and special relativity beyond the usual "they agree in the limit for slow speeds". To compare theories we use techniques, such as logical interpretation and definitional equivalence, from definability theory. Those are usually used to show that theories are equivalent, here we use them to pinpoint the exact differences between both theories by showing how the theories need to be changed to make them equivalent.

To achieve that, both theories have been axiomatized within *many-sorted first order logic with equivalence*, in the spirit of the algebraic logic approach of the Andréka–Németi school, for instance in [1], [3], [4], [5], [6], [30] and [45]. Our axiom system for special relativity is one of the many slightly different variants of SpecRel. The main differences from stock SpecRel are firstly that all other versions of SpecRel use the lightspeed $c = 1$ (which has the advantage of simpler formulas and calculations), while we have chosen to make our results more general by not assuming any units in which to measure the speed of light; and secondly we have chosen to fill the models with all the potential inertial observers by including axioms AxThExp and AxTriv, which already exists in [1, p.135] and [30, p.81], but which is not included in the axiom system in most of the literature.

There also already exists NewtK as a set of axioms for classical kinematics in [1, p.426], but that has an infinite speed of light, which is well-suited to model "early" classical kinematics before the discovery of the speed of light in 1676 by O. Rømer, while we target "late" classical kinematics, more specifically in the nineteenth century at the time of J. C. Maxwell and the search for the *luminiferous ether*.

The epistemological significance of our school's research project in general and the the kind of research done in current dissertation in particular is being discussed in [23].

An advantage of using first order logic is that it forces us to reveal all the tacit assumptions and formulate explicit formulas with clear and unambiguous meaning. Another one is that it would make it easier to validate our proofs by machine verification, see [42] and [43]. For the precise definition of the syntax and semantics of first order logic, see, e.g., [17, §1.3], [21, §2.1, §2.2].

In its spirit relativity theory has always been axiomatic since its birth, as in

1905 A. Einstein introduced special relativity by two informal postulates in [20]. This original informal axiomatization was soon followed by formal ones, starting with A. A. Robbs [40], many others, for example in [9], [14], [24], [25], [35], [38], [41], [47] and [51] several of which are still being investigated. For example, the historical axiom system of J. Ax which uses simple primitive concepts but a lot of axioms to axiomatize special relativity has been proven in [7] to be definitional equivalent to a variant of the Andréka–Németi axioms which use only four axioms but more complex primitive notions.

Our use of techniques such as *logical interpretation* and *definitional equivalence* can be situated among a wider interest and study of these concepts currently going on, for example in [11], [12], [28] and [49]. Definitional equivalence has also been called *logical synonymity* or *synonymy*, for example in [16], [18] and [22]. The first known use of the method of definitional equivalence is, according to [19], in [34].

Our approach of using Poincaré–Einstein synchronisation in classical mechanics was inspired by the "sound model for relativity" of [9]. We were also inspired by L. E. Szabó's paper [44] claiming that Einstein's main contribution was redefining the basic concepts of time and space in special relativity.

Let us now formally introduce the concepts *translation*, *interpretation* and *definitional equivalence* and present our main results:

Definition 1. A *translation* Tr is a function between formulas of many-sorted languages having the same sorts which

- translates any n-ary relation[1] R into a formula having n free variables of the corresponding sorts: $Tr[R(x_1 \dots x_n)] \equiv \varphi(x_1 \dots x_n)$,
- preserves the equality for every sort, i.e. $Tr(v_i = v_j) \equiv v_i = v_j$,
- preserves the quantifiers for every sort, i.e. $Tr[(\forall v_i)(\varphi)] \equiv (\forall v_i)[Tr(\varphi)]$ and $Tr[(\exists v_i)(\varphi)] \equiv (\exists v_i)[Tr(\varphi)]$,
- preserves complex formulas composed by logical connectives, i.e. $Tr(\neg\varphi) \equiv \neg Tr(\varphi)$, $Tr(\varphi \wedge \psi) \equiv Tr(\varphi) \wedge Tr(\psi)$, etc.

Definition 2. An *interpretation* of theory Th_1 in theory Th_2 is a translation Tr which translates all axioms(and hence all theorems) of Th_1 into theorems of Th_2:

$$(\forall\varphi)[Th_1 \vdash \varphi \Rightarrow Th_2 \vdash Tr(\varphi)].$$

There are several definitions for *definitional equivalence*, e.g. [8, p. 39-40, §4.2], [11, p. 469-470] [30, p. 42] [27, pp. 60-61], and [48, p. 42], which are all equivalent

[1] In the definition we concentrate only on the translation of relations because functions and constants can be reduced to relations, see e.g. [13, p. 97 §10].

if the languages of the theories have disjoint vocabularies. Our definition below is a syntactic version of the semantic definition in [26, p. 56, §0.1.6]:

Definition 3. An interpretation Tr of Th_1 in Th_2 is a *definitional equivalence* if there is another interpretation Tr' such that the following holds for every formula φ and ψ of the corresponding languages:

- $Th_1 \vdash Tr'\big(Tr(\varphi)\big) \leftrightarrow \varphi$
- $Th_2 \vdash Tr\big(Tr'(\psi)\big) \leftrightarrow \psi$

We denote the definitional equivalence of Th_1 and Th_2 by $Th_1 \equiv_\Delta Th_2$.

Definition 4. By a the translation of a set of formulas Th, we mean the set of the translations of all formulas in the set Th:

$$Tr(Th) \stackrel{\text{def}}{=} \{Tr(\varphi) : \varphi \in Th\}.$$

Theorem 1. Definitional equivalence is an equivalence relation, i.e. it is reflexive, symmetric and transitive.

Proof.

- Definitional equivalence is reflexive: Let the translation functions Tr and its inverse Tr' both be the identity Id which translates theory Th into itself: $Tr(Th) = Th$ and $Tr'(Th) = Th$, therefor $Th \equiv_\Delta Th$.
- Definitional equivalence is symmetric by its definition.
- Definitional equivalence is transitive: Since interpretations translate the theorems of one theory into theorems of another theory, the composition of interpretations is an interpretation too. If Tr is an interpretation with inverse Tr' showing the equivalence of Th_1 and Th_2 and TR is an interpretation with inverse TR' showing the equivalence of Th_2 and Th_3, then $x \mapsto TR(Tr(x))$ is an interpretation with inverse $x \mapsto Tr'(TR'(x))$ showing the equivalence of Th_1 and Th_3. □

In this dissertation, we introduce axiom systems $\mathsf{ClassicalKin}_{\mathsf{Full}}$ for classical kinematics, $\mathsf{SpecRel}_{\mathsf{Full}}$ for special relativity and their variants based on the framework and axiom system of [1], [3], [4], [5], [6] and [30]. Then we construct logical interpretations between these theories translating the axioms of one system into theorems of the other. In more detail we show the following connections:

Special relativity can be interpreted by classical kinematics, i.e., there is a translation Tr that translates the axioms of special relativity into theorems of classical kinematics:

- $\mathsf{ClassicalKin}_{\mathsf{Full}} \vdash Tr(\mathsf{SpecRel}_{\mathsf{Full}})$. [see Theorem 4 on p.38]

Special relativity extended with a concept of ether $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ and classical kinematics restricted to slower-than-light observers $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ can be interpreted by each other:

- $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr_{+}(\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}})$ [see Theorem 6 on p.48]
- $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_{+}(\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}})$. [see Theorem 7 on p.52]

Moreover, these axiom systems are definitionally equivalent ones:

- $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \equiv_{\Delta} \mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$. [see Theorem 8 on p.56]

Furthermore, we establish the definitional equivalence between $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ and $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$, which is identical to $\mathsf{ClassicalKin}_{\mathsf{Full}}$ except that the existence of non-inertial observers is explicitely forbidden:

- $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}} \equiv_{\Delta} \mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ [see Theorem 11 on p.71]

from which follows, by transitivity of definitional equivalence, that classical kinematics is definitional equivalent to special relativity extended with ether:

- $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}} \equiv_{\Delta} \mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$, [see Corollary 9 on p. 72]

which is the main result of this dissertation.

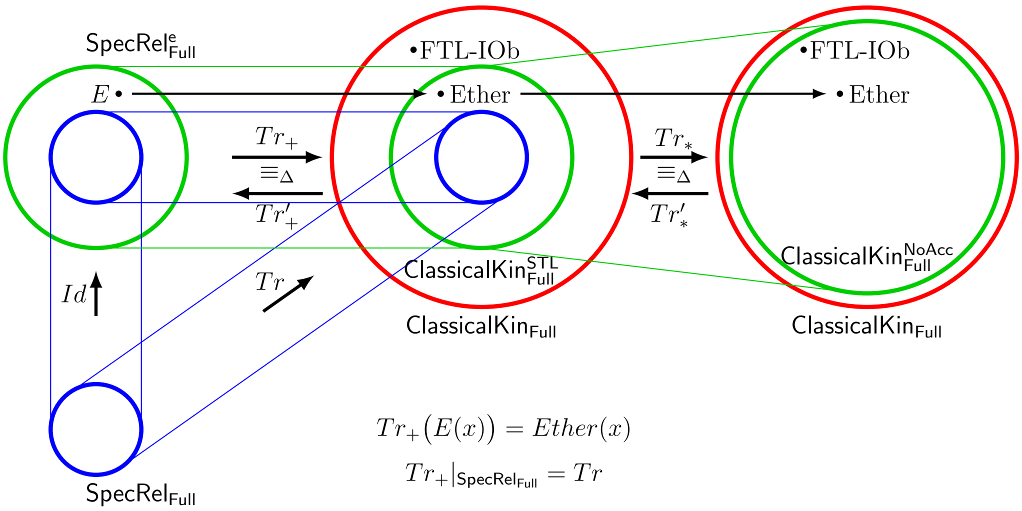

Fig. 1.1: Translations: Tr translates from special relativity to classical kinematics. Tr_+ and Tr'_+ translate between special relativity extended with a primitive ether E (which is just a distinguished class of observers which are stationary relative to each other) and classical kinematics without faster-than-light observers, which are definitionally equivalent theories, see Theorem 8. Tr_* and Tr'_* translate between classical kinematics without faster-than-light observers and classical kinematics, which are definitionally equivalent theories, see Theorem 11.

2. THE LANGUAGES OF OUR THEORIES

We will work in the axiomatic framework of [5].

We will use two separate languages for classical kinematics (with the superscript CK) and special relativity (with the superscript SR). When there is no confusion possible or when we talk about properties which are true for both languages, we will ommit the superscrips.

Therefore, there will be two sorts of basic objects: *bodies* B (thing that can move) and *quantities* Q (numbers used by observers to describe motion via coordinate systems). We will distinguish two kinds of bodies *inertial observers* and *light signals* by one-place relation symbols IOb and Ph of sort B. We will use the usual algebraic operations and ordering ($+$, $\cdot$ and $\leq$) on sort Q. Finally, we will formulate coordinatization by using a 6-place *worldview relation* W of sort $B^2 \times Q^4$.

That is, we will use the following two-sorted first order logics with equality:

$$\{\, B, Q\,; IOb^{CK}, Ph^{CK}, +^{CK}, \cdot^{CK}, \leq^{CK}, W^{CK}\,\}$$

for classical kinematics and

$$\{\, B, Q\,; IOb^{SR}, Ph^{SR}, +^{SR}, \cdot^{SR}, \leq^{SR}, W^{SR}\,\}$$

for special relativity theory.

Relations $IOb(k)$ and $Ph(p)$ are translated as "*k is an inertial observer*," and "*p is a light signal*," respectively. $W(k, b, x_0, x_1, x_2, x_3)$ is translated as "*body k coordinatizes body b at space-time location $\langle x_0, x_1, x_2, x_3\rangle$*," (i.e., at space location $\langle x_1, x_2, x_3\rangle$ and instant x_0).

Since we have two sorts (quantities and bodies), we have also two kinds of variables, two kinds of terms, two equation signs and two kinds of quantifiers (one for each corresponding sort). Quantity variables are usually denoted by x, y, z, t, v, c (and their indexed versions), body variables are usually denoted by b, k, h, e, p (and their indexed versions). Since we have no function symbols of sort B, *body terms* are just the body variables. *Quantity terms* are what can be built from quantity variables using the two functions symbols $+$ and $\cdot$ of sort Q. We denote quantity terms by α, β, γ (and their indexed versions). For convenience, we use the same sign ($=$) for both sorts because from the context it is always clear whether we mean equation between quantities or bodies.

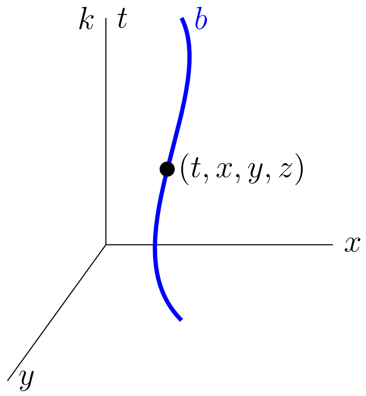

Fig. 2.1: $W(k, b, t, x, y, z)$: Observer k coordinatizes body b at position (t, x, y, z).

The so called *atomic formulas* of our language are $W(k, b, \alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $IOb(k)$, $Ph(p)$, $\alpha = \beta$, $\alpha \leq \beta$ and $k = b$ where k, p, b, α, β, α_0, α_1, α_2, α_3 are arbitrary terms of the corresponding sorts.

The *formulas* are built up from these atomic formulas by using the logical connectives *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$), *if-and-only-if* ($\leftrightarrow$) and the quantifiers *exists* ($\exists$) and *for all* ($\forall$). In long expressions, we will denote the logical *and* by writing formulas one below the other between rectangular brackets:

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix} \text{ is a notation for } \varphi \wedge \psi.$$

To distinguish formulas *in* our language from formulas *about* our language, we use in the meta-language the symbols $\Leftrightarrow$ (as illustrated in the definition of bounded quantifiers below) and $\equiv$ (while translating formulas between languages) for the logical equivalence. We use the symbol $\vdash$ for syntactic consequence.

We use the notation Q^n for the set of n-tuples of Q. If $\bar{x} \in Q^n$, we assume that $\bar{x} = \langle x_1, \ldots, x_n \rangle$, i.e., x_i denotes the i-th component of the n-tuple $\bar{x}$. We also write $W(k, b, \bar{x})$ in place of $W(k, b, x_0, x_1, x_2, x_3)$, etc.

We will treat unary relations as sets. If R is a unary relation, then we use bounded quantifiers in the following way:

$$(\forall u \in R)[\varphi] \stackrel{\text{def}}{\iff} \forall u[R(u) \rightarrow \varphi]$$
$$(\exists u \in R)[\varphi] \stackrel{\text{def}}{\iff} \exists u[R(u) \wedge \varphi]$$

We will also use bounded quantifiers to make it explicit which the sort of the variable is, such as $\exists x \in Q$ and $\forall b \in B$, to make our formulas easier to comprehend.

Worldlines and events can be easily expressed by the worldview relation W as follows.

Definition 5. The *worldline* of body b according to observer k is the set of coordinate points where k have coordinatized b:

$$\bar{x} \in wl_k(b) \stackrel{\text{def}}{\iff} W(k, b, \bar{x}).$$

Definition 6. The *event* occurring for observer k at coordinate point $\bar{x}$ is the set of bodies k observes at $\bar{x}$:

$$b \in ev_k(\bar{x}) \stackrel{\text{def}}{\iff} W(k, b, \bar{x}).$$

We will use a couple of shorthand notations to discuss spatial distance, speed, etc.[2]

Definition 7. *spatial distance* of $\bar{x}, \bar{y} \in Q^4$:

$$space(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Definition 8. *time difference* of coordinate points $\bar{x}, \bar{y} \in Q^4$:

$$time(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} |x_0 - y_0|.$$

Definition 9. The *speed* of body b according to observer k

$$speed_k(b) = v \stackrel{\text{def}}{\iff} \big(\exists \bar{x}, \bar{y} \in wl_k(b)\big)(\bar{x} \neq \bar{y}) \land \big(\forall \bar{x}, \bar{y} \in wl_k(b)\big) \left[space(\bar{x}, \bar{y}) = v \cdot time(\bar{x}, \bar{y})\right].$$

Speed is a partial function from $B \times B$ to Q which is defined if $wl_k(b)$ is a subset of the non-horizontal line which contains at least two points.

Definition 10. The *velocity* of body b according to observer k

$$\bar{v}_k(b) = \bar{v} \stackrel{\text{def}}{\iff} \big(\exists \bar{x}, \bar{y} \in wl_k(b)\big)(\bar{x} \neq \bar{y}) \land \big(\forall \bar{x}, \bar{y} \in wl_k(b)\big) \left[(y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0)\right].$$

Velocity is a partial function from $B \times B$ to Q^3 which is defined if $wl_k(b)$ is a subset of the non-horizontal line which contains at least two points.

Definition 11. The *worldview transformation*[3] between observers k and k' is the following binary relation on Q^4:

$$w_{kk'}(\bar{x}, \bar{y}) \stackrel{\text{def}}{\iff} ev_k(\bar{x}) = ev_{k'}(\bar{y}).$$

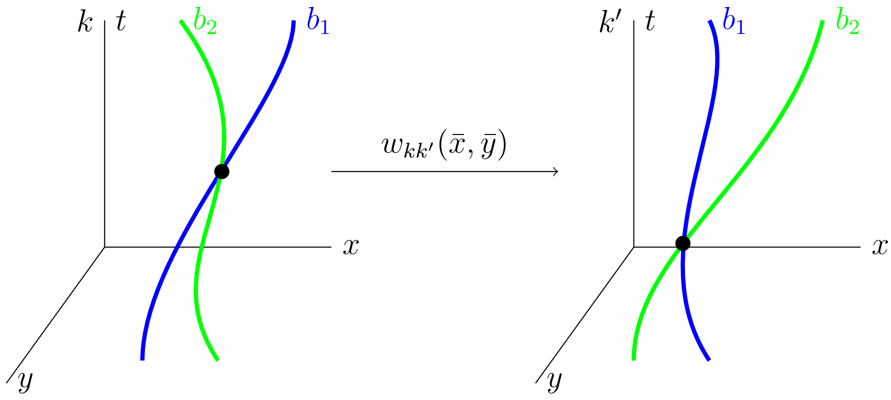

Fig. 2.2: Worldview transformation from observer k to observer k'.

Convention 1. We use partial functions in our formulas as special relations, which means that when we write $f(x)$ in a formula we also assume that $f(x)$ is defined i.e., x is in the domain of f. This is only a notational convention that makes our formulas more human readable and can be systematically eliminated, see [1, p. 61, Convention 2.3.10] for further discussion.

The *models* of this language are of the form

$$\mathfrak{M} = \langle B_{\mathfrak{M}}, Q_{\mathfrak{M}}; IOb_{\mathfrak{M}}, Ph_{\mathfrak{M}}, +_{\mathfrak{M}}, \cdot_{\mathfrak{M}}, \leq_{\mathfrak{M}}, W_{\mathfrak{M}} \rangle,$$

where $B_{\mathfrak{M}}$ and $Q_{\mathfrak{M}}$ are nonempty sets, $IOb_{\mathfrak{M}}$ and $Ph_{\mathfrak{M}}$ are unary relations on $B_{\mathfrak{M}}$, $+_{\mathfrak{M}}$ and $\cdot_{\mathfrak{M}}$ are binary functions and $\leq_{\mathfrak{M}}$ is a binary relation on $Q_{\mathfrak{M}}$, and $W_{\mathfrak{M}}$ is a relation on $B_{\mathfrak{M}} \times B_{\mathfrak{M}} \times Q_{\mathfrak{M}}^d$. The subscript $_{\mathfrak{M}}$ for the sets and relations indicates that those are set theoretical objects rather than the symbols of a formal language.

[2] Since in our language we only have addition and multiplication, we need some basic assumptions on the properties of these operators on numbers ensuring the definability of subtraction, division, and square roots. These properties will follow from the Euclidian field axiom (AxEField, below on page 14). Also, the definition of speed is based on the axiom saying that inertial observers move along straight lines relative to each other (AxLine, below on page 14).

[3] While worldview transformation w is here only defined as a binary relation, our axioms will turn it into a transformation for inertial observers, see Theorems 2 and 3 below.

3. AXIOMS

3.1 Axioms for the common part

For the structure $\langle Q, +, \cdot, \leq \rangle$ of quantities, we assume some basic algebraic properties of addition, multiplication and ordering true for real numbers.

<u>AxEField</u> $\langle Q, +, \cdot, \leq \rangle$ is an Euclidean field. That is, $\langle Q, +, \cdot \rangle$ is a field in the sense of algebra; $\leq$ is a linear ordering on Q such that $x \leq y \to x + z \leq y + z$, and $0 \leq x \land 0 \leq y \to 0 \leq xy$; and every positive number has a square root.

Some notable examples of Euclidean fields are the real numbers, the real algebraic numbers, the hyperreal numbers and the real constructable numbers[4].

The rest of our axioms will speak about how inertial observers coordinatize the events.

Naturally, we assume that they coordinatize the same set of events.

<u>AxEv</u> All inertial observers coordinatize the same events:

$$(\forall k, h \in IOb)\big(\forall \bar{x} \in Q^4\big)\big(\exists \bar{y} \in Q^4\big)[ev_k(\bar{x}) = ev_h(\bar{y})].$$

We assume that inertial observers move along straight lines with respect to each other.

<u>AxLine</u> The worldline of an inertial observer is a straight line according to inertial observers:

$$(\forall k, h \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{z} \in wl_k(h)\big)$$
$$(\exists a \in Q)\big[\bar{z} - \bar{x} = a(\bar{y} - \bar{x}) \lor \bar{y} - \bar{z} = a(\bar{z} - \bar{x})\big].$$

As usual we speak about the motion of reference frames by using their time-axises. Therefore, we assume the following.

<u>AxSelf</u> Any inertial observer is stationary in his own coordinate system:

$$(\forall k \in IOb)(\forall t, x, y, z \in Q)\big[W(k, k, t, x, y, z) \leftrightarrow x = y = z = 0\big].$$

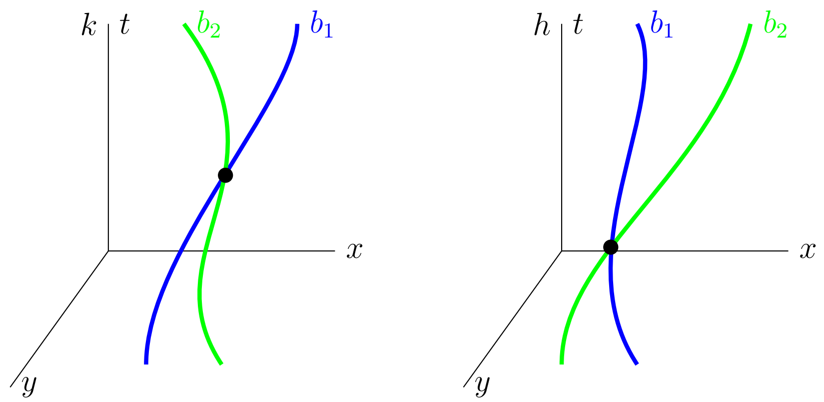

Fig. 3.1: AxEv: Inertial observers coordinatize the same events (meetings of bodies).

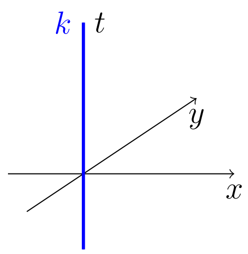

Fig. 3.2: AxSelf: Every Inertial observer is stationary according to himself.

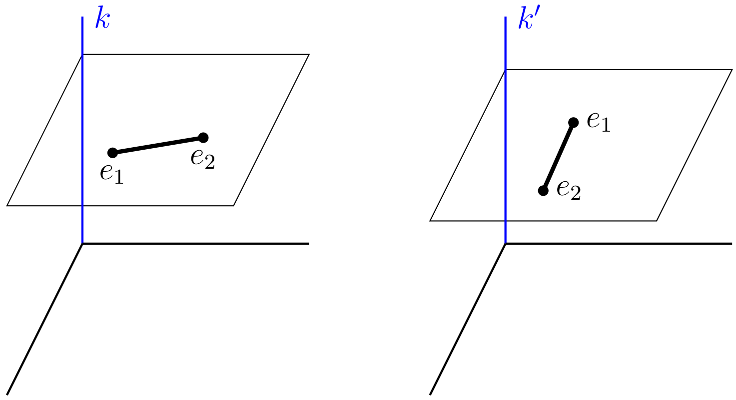

Fig. 3.3: AxSymD: Inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them.

The following axiom is a symmetry axiom saying that observers (can) use the same units to measure spatial distances.

AxSymD Any two inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them:

$$(\forall k, k' \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4\big)$$
$$\left(\begin{bmatrix} time(\bar{x},\bar{y}) = time(\bar{x}',\bar{y}') = 0 \\ ev_k(\bar{x}) = ev_{k'}(\bar{x}') \\ ev_k(\bar{y}) = ev_{k'}(\bar{y}') \end{bmatrix} \rightarrow space(\bar{x},\bar{y}) = space(\bar{x}',\bar{y}')\right).$$

When we choose an inertial observer to represent an inertial frame[5] of reference, then the origin of that observer can be chosen anywhere, as well as the orthonormal basis they use to coordinatize space. To introduce an axiom capturing this idea, let *Triv* be the set of *trivial transformations*, by which we simply mean transformations that are isometries on space and translations along the time axis. For more details, see [30, p. 81].

AxTriv Any trivial transformation of an inertial coordinate system is also an inertial coordinate system:

$$(\forall T \in Triv)(\forall k \in IOb)(\exists k' \in IOb)[w_{kk'} = T].^6$$

[4] It is an open question if the rational numbers would be sufficient, which would allow to replace AxEField by the weaker axiom that the quantities only have to be an ordered field, and hence have a stronger result since we would be assuming less. See [31] for a possible approach.

[5] We use the word "frame" here in its intuitive meaning, as in [39, p.40]. For a formal definition of a *frame* we need the concept of *trivial transformation*, see the next page.

[6] $(\forall T \in Triv)$ may appear to the reader to be in second order logic. However, since a trivial

Axioms AxTriv, $\mathsf{AxThExp}_+$ (on page 17), and AxThExp (on page 23) make spacetime full of inertial observers.[7] We will use the subscript Full to denote that these axioms are part of our axiom systems.

A set of all observers which are at rest relative to each other, which is a set of all observers which are related to each other by a trivial transformation, we call a *frame*. Because of AxTriv, a frame contains an infinite number of elements. From now on, we may informally abbreviate "the speed/velocity/movement relative to all observers which are elements of a frame" to "the speed/velocity/movement relative to a frame".

The axioms above will be part of all the axiom systems that we are going to use in this dissertation. Let us call their collection $\mathsf{Kin}_{\mathsf{Full}}$:

$$\mathsf{Kin}_{\mathsf{Full}} \stackrel{\text{def}}{=} \{\mathsf{AxEField}, \mathsf{AxEv}, \mathsf{AxSelf}, \mathsf{AxSymD}, \mathsf{AxLine}, \mathsf{AxTriv}\}.$$

3.2 Axioms for classical kinematics

A key assumption of classical kinematics is that the time difference between two events is observer independent.

<u>AxAbsTime</u> The time difference between any two events is the same for all inertial observers:

$$(\forall k, k' \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4\big) \left(\begin{bmatrix} ev_k(\bar{x}) = ev_{k'}(\bar{x}') \\ ev_k(\bar{y}) = ev_{k'}(\bar{y}') \end{bmatrix} \rightarrow time(\bar{x}, \bar{y}) = time(\bar{x}', \bar{y}') \right).$$

We also assume that inertial observers can move with arbitrary (finite) speed in any direction everywhere.

<u>$\mathsf{AxThExp}_+$</u> Inertial observers can move along any non-horizontal straight line[8]:

$$(\exists h \in B)\big[IOb(h)\big]\wedge (\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big) \left(x_0 \neq y_0 \rightarrow \big(\exists k' \in IOb\big)\big[\bar{x}, \bar{y} \in wl_k(k')\big]\right).$$

transformation is nothing but an isometry on space (4 × 4 parameters) and a translation along the time axis (4 parameters), this is just an abbreviation for $(\forall q_1, q_2, \ldots q_{20} \in Q)$ and together with $w_{kk'} = T$ a system of equations with 20 parameters in first order logic.

[7] These inertial observers are only potential and not actual observers. The same way as the light signals required in every coordinate points by axioms $\mathsf{AxPh_c}$ and AxEther below are only potential light signals. For a discussion on how actual and potential bodies can be distinguished using modal logic, see [33].

[8] The first part of this axiom (before the conjumction) is not necessary since we will asume the axiom AxEther which guarantees that we have at least one inertial observer, see page 19.

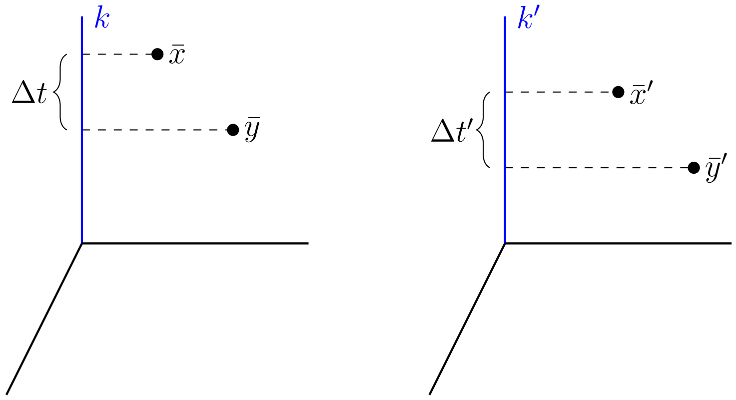

Fig. 3.4: AxAbsTime: Time is absolute, the time difference $\Delta t = \Delta t'$ between two events is the same for all observers.

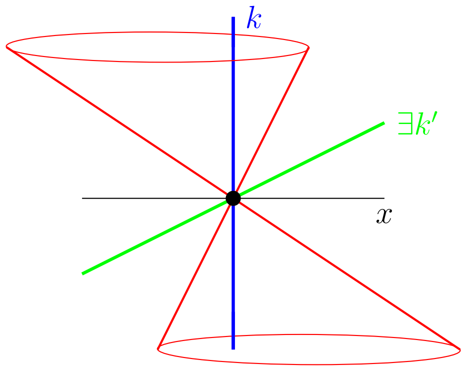

Fig. 3.5: $\mathsf{AxThExp}_+$: Inertial observers can move along any non-horizontal straight line.

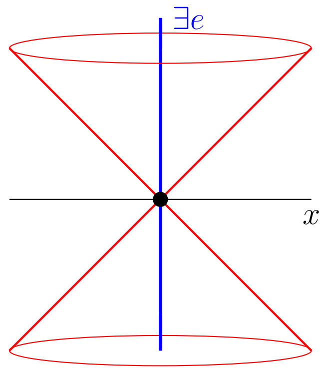

Fig. 3.6: AxEther: There exists an inertial observer in which the light cones are right.

The motion of light signals in classical kinematics is captured by assuming that there is at least one inertial observer according to which the speed of light is the same in every direction everywhere. Inertial observers with this property will be called *ether observers* and the unary relation *Ether* appointing them is defined as follows:

Definition 12.

$$Ether(e) \stackrel{\text{def}}{\iff} IOb(e) \land (\exists c \in Q)\Big[c > 0 \land \big(\forall \bar{x}, \bar{y} \in Q^4\big) \\ \Big((\exists p \in Ph)\big[\bar{x}, \bar{y} \in wl_e(p)\big] \leftrightarrow space(\bar{x}, \bar{y}) = c \cdot time(\bar{x}, \bar{y})\Big)\Big].$$

<u>AxEther</u> There exists at least one ether observer:

$$(\exists e \in B)\big[Ether(e)\big].$$

Let us introduce the following axiom system for classical kinematics:

$$\mathsf{ClassicalKin_{Full}} \stackrel{\text{def}}{=} \mathsf{Kin_{Full}} \cup \{\mathsf{AbsTime}, \mathsf{AxThExp_+}, \mathsf{AxEther}\}.$$

Definition 13. Map $G : Q^4 \to Q^4$ is called a *Galilean transformation* iff it is an affine bijection having the following properties

$$|time(\bar{x}, \bar{y})| = |time(\bar{x}', \bar{y}')|$$

and

$$x_0 = y_0 \to x'_0 = y'_0 \land space(\bar{x}, \bar{y}) = space(\bar{x}', \bar{y}')$$

for all $\bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4$ for which $G(\bar{x}) = \bar{x}'$ and $G(\bar{y}) = \bar{y}'$.

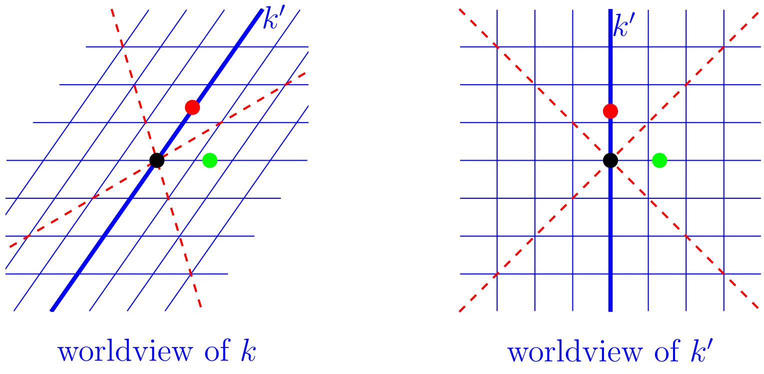

Fig. 3.7: In $\mathsf{ClassicalKin_{Full}}$, worldview transformations are Galilean transformations. (In the figures, we asume that the speed of light is 1.)

We now prove the *justification theorem*, to establish that the above is indeed an axiomatization of classical kinematics:

Theorem 2. Assume $\mathsf{ClassicalKin_{Full}}$. Then w_{mk} is a Galilean Transformation for all inertial observers m and k.

Proof. Let m and k be inertial observes. For all $\bar{x}, \bar{y} \in Q^4$ and inertial observer h, there is an inertial observer b such that $b \in ev_h(\bar{x})$ but $b \notin ev_h(\bar{y})$ by axioms AxEField, AxThExp$_+$ and AxLine. Therefore, binary relations w_{mk} and $w_{km} = (w_{mk})^{-1}$ are injective functions. Hence w_{mk} is a bijection of Q^4.

By axiom AxAbsTime, w_{mk} takes horizontal hyperplanes to horizontal ones. By AxLine, worldlines of inertial observers are lines. Clearly, w_{mk} takes worldlines of inertial observers to worldlines of inertial observers. Therefore w_{mk} takes non-horizontal lines to non-horizontal ones.

Let us now show that w_{mk} also takes non-horizontal planes to non-horizontal planes. To do so, let P be such a plane. Let a and b be two inertial observers such that lines $wl_m(a)$ and $wl_m(b)$ are distinct and subsets of P (by AxThExp$_+$ there are plenty of such observers). Let $\bar{y}$ be the intersection of $wl_m(a)$ and $wl_m(b)$. Let $\bar{x}$ be an arbitrary point of P which is not on $wl_m(a)$ or $wl_m(b)$. Since w_{mk} is a bijection, we know that lines $wl_k(a)$ and $wl_k(b)$ are also distinct. Let P' be the plane determined by lines $wl_k(a)$ and $wl_k(b)$. Let $\bar{x}'$ be the w_{mk} image of $\bar{x}$. We are going to show that $\bar{x}' \in P'$. To do so, let c and d be inertial observers such that $\bar{x} \in wl_m(c) \cap wl_m(d)$ and lines $wl_m(c)$ and $wl_m(d)$ intersect lines $wl_m(a)$ and $wl_m(b)$ in distinct points $\bar{z}$, $\bar{u}$, $\bar{v}$, and $\bar{w}$, see Figure 3.8. There are such observers by AxThExp$_+$.

Since w_{mk} is a bijection, lines $wl_k(a)$, $wl_k(b)$, $wl_k(c)$, and $wl_k(d)$ intersect each other in different points. Let $\bar{y}'$, $\bar{z}'$, $\bar{u}'$, $\bar{v}'$, and $\bar{w}'$ be the w_{mk} images the respective

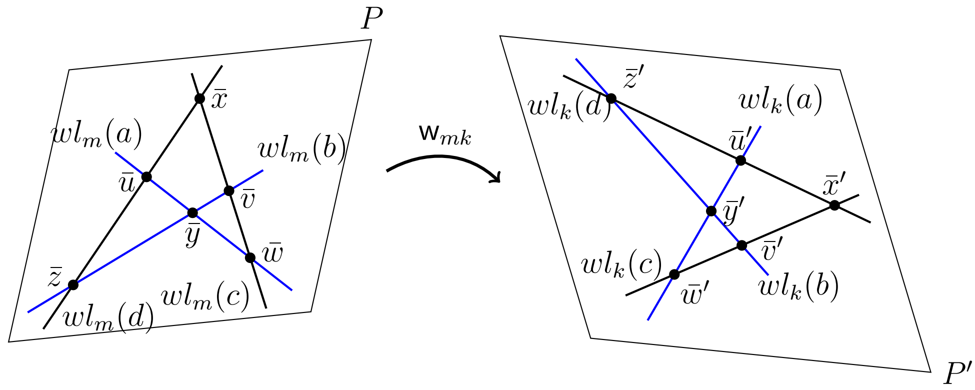

Fig. 3.8: Justification Theorem for $\mathsf{ClassicalKin}_{\mathsf{Full}}$: Theorem 2

coordinate points. Since triangle $\bar{v}'\bar{w}'\bar{y}'$ is non-degenerate, $wl_k(c)$ is in P'. Similarly, $wl_k(d)$ is also in P'. Therefore, their intersection $\bar{x}'$ is also in P'. Therefore, every point $\bar{x}$ of P which is not on $wl_m(a) \cup wl_m(b)$ is mapped to P' by w_{mk}. It is clear that w_{mk} images of points of $wl_m(a) \cup wl_m(b)$ are also in P'. Therefore, every point of plane P is mapped into plane P'; and this is what we wanted to show.

Now we are going to show that w_{mk} also takes horizontal lines to (horizontal) lines. To do so, let l be a horizontal line. There are non-horizontal planes P_1 and P_2 such that $l = P_1 \cap P_2$. We have already seen that w_{mk} takes P_1 and P_2 to non-horizontal planes. Let us denote these planes P_1' and P_2'. Since w_{mk} is a bijection, $P_1' \cap P_2'$ is also a line and w_{mk} has to take l to this line. Therefore, we have shown that w_{mk} is a bijection of Q^4 taking lines to lines. Hence w_{mk} is a collineation of Q^4. Consequently, by the fundamental theorem of affine geometry[9], w_{mk} is a semi-affine transformation. By AxSymD, w_{mk} have to be an isometry restricted to horizontal planes. Hence w_{mk} has to be an affine transformation.

So we have that w_{mk} is an affine bijection. To finish our proof, we have to check the two extra properties of Galilean transformations. Let $\bar{x}$ and $\bar{y}$ be two coordinate points and let $\bar{x}'$ and $\bar{y}'$ be their respective w_{mk} images. By AxAbsTime, we have that

$$|time(\bar{x}, \bar{y})| = |time(\bar{x}', \bar{y}')|;$$

and since w_{mk} maps horizontal hyperplanes isometrically to horizontal hyperplanes,

[9] Fundamental theorem of affine geometry: Let X, X' be affine spaces of same finite dimension $d \geq 2$, let $f : X \to X'$ be a bijection which takes any three collinear points $a, b, c \in X$ into collinear points $f(a), f(b), f(c) \in X'$, then f is semi-affine [15, p. 52, §2.6.3]. In our case, a semi-affine map f from Q^4 to Q^4 is an affine map composed by a field automorphism induced bijection, which is a bijection generated by applying a field automorphism to all 4 coordinates t, x, y and z.

we also have that

$$x_0 = y_0 \to x_0 = y_0' \wedge space(\bar{x}, \bar{y}) = space(\bar{x}', \bar{y}');$$

and this is what we wanted to prove. □

Theorem 2 shows that $\mathsf{ClassicalKin_{Full}}$ captures classical kinematics since it implies that the worldview transformations between inertial observers are the same as in the standard non-axiomatic approaches.

There is a similar theorem as Theorem 2 for NewtK, a version of classical kinematics with $c = \infty$, in [1, p.439, Proposition 4.1.12 Item 3].

Corollary 1. Assuming $\mathsf{ClassicalKin_{Full}}$, all ether observers are stationary with respect to each other, and hence they agree on the speed of light.

Proof. Assuming $\mathsf{ClassicalKin_{Full}}$, let e_1 and let e_2 be ether observers, and p_1 and p_2 be light signals. Light cones are the set of worldlines of light signals through a given point. By the definition of ether observers, light cones are symmetric cones for all ether observers. By item 1 in Theorem 2 below, the worldview transformation between e_1 and e_2 is a Galilean transformation. Observers e_1 and e_2 see each other's light cones right because worldlines of light signals are mapped on worldlines of light signals. This is only possible if the Galilean transformations between observers e_1 and e_2 are trivial, meaning that they are stationary with respect to each other. Hence $speed_{e_1}(p_1) = speed_{e_2}(p_2)$. Since e_1 and e_2 are chosen arbitrarily, this holds for all ether observers. □

Since *Ether* is an unary relation, we can also treat it as a set. So, by Corollary 1, *Ether* as a set is the *ether frame* and its elements (usually denoted by e_1, e_2, e_3, ... or e, e', e'', ...) are the *ether observers*. The way in which we distinguish *frames* from *observers* is inspired by W. Rindler in [39, p.40].

By Corollary 1, we can speak about the *ether-observer-independent speed of light*, denoted by $\mathfrak{c}_\mathfrak{e}$ which is the unique quantity satisfying the following formula:

$$(\forall e \in Ether)(\forall p \in Ph)[speed_e(p) = \mathfrak{c}_\mathfrak{e}].$$

Corollary 2. Assuming $\mathsf{ClassicalKin_{Full}}$, the speed of any inertial observer is the same according to all ether observers:

$$(\forall e, e' \in Ether)(\forall k \in IOb)[speed_e(k) = speed_{e'}(k)].$$

Corollary 3. Assuming $\mathsf{ClassicalKin_{Full}}$, all ether observers have the same velocity according to any inertial observer:

$$(\forall e, e' \in Ether)(\forall k \in IOb)[\bar{v}_k(e) = \bar{v}_k(e')].$$

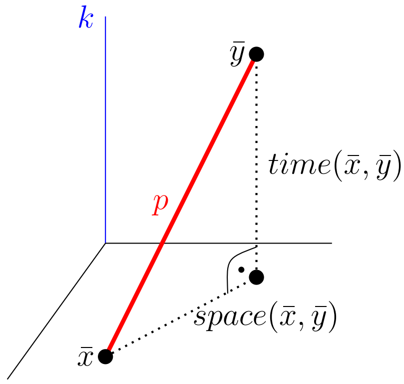

Fig. 3.9: $\mathsf{AxPh_c}$: For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.

3.3 Axioms for special relativity

The key assumption of special relativity is that the speed of light signals is observer independent.

$\underline{\mathsf{AxPh_c}}$ For any inertial observer, the speed of light is the same everywhere and in every direction. Furthermore, it is possible to send out a light signal in any direction everywhere:

$$(\exists c \in Q)\Big[c > 0 \land (\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big)$$
$$\big((\exists p \in Ph)\big[\bar{x}, \bar{y} \in wl_k(p)\big] \leftrightarrow space(\bar{x}, \bar{y}) = c \cdot time(\bar{x}, \bar{y})\big)\Big].$$

By $\mathsf{AxPh_c}$, we have an observer-independent speed of light. From now on, we will denote this speed of light as $\mathfrak{c}$. From $\mathsf{AxPh_c}$ follows that observers (as considered by the theory) use units of measurement where they have the same numerical value for the speed of light. The value of the constant speed of light depends on the choice of units (for example $\mathfrak{c} = 299792458$ when using meters and seconds or $\mathfrak{c} = 1$ when using light-years and years as units). We prove below, in Lemma 2 and Corollary 6, that the relativistic speed of light $\mathfrak{c}$ and the ether-observer-independent speed of light $\mathfrak{c}_\mathfrak{e}$ translate into each other. Note that c in AxPh and $\mathsf{AxEther}$ is a variable, while $\mathfrak{c}$ and $\mathfrak{c}_\mathfrak{e}$ are model dependent constants.

$\mathsf{Kin_{Full}}$ and $\mathsf{AxPh_c}$ imply that no inertial observer can move faster than light if $d \geq 3$, see, e.g., [5]. Therefore, we will use the following version of $\mathsf{AxThExp_+}$.

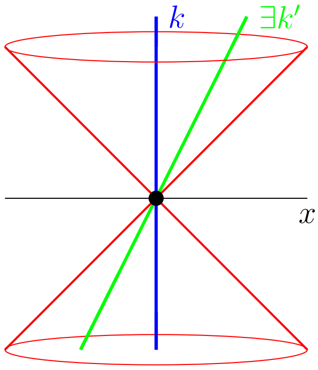

Fig. 3.10: AxThExp: Inertial observers can move with any speed slower than that of light.

AxThExp Inertial observers can move along any straight line of any speed less than the speed of light:

$$(\exists h \in B)\big[IOb(h)\big] \wedge (\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big)\big(space(\bar{x}, \bar{y}) < \mathfrak{c} \cdot time(\bar{x}, \bar{y}) \rightarrow (\exists k' \in IOb)\big[\bar{x}, \bar{y} \in wl_k(k')\big]\big).$$

Let us introduce the following axiom system for special relativity:

$$\mathsf{SpecRel}_{\mathsf{Full}} \stackrel{\text{def}}{=} \mathsf{Kin}_{\mathsf{Full}} \cup \{\mathsf{AxPh}_{\mathsf{c}}, \mathsf{AxThExp}\}.$$

Map $P : Q^4 \rightarrow Q^4$ is called a *Poincaré transformation* corresponding to light speed c iff it is an affine bijection having the following property

$$c^2 \cdot time(\bar{x}, \bar{y})^2 - space(\bar{x}, \bar{y})^2 = c^2 \cdot time(\bar{x}', \bar{y}')^2 - space(\bar{x}', \bar{y}')^2$$

for all $\bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4$ for which $P(\bar{x}) = \bar{x}'$ and $P(\bar{y}) = \bar{y}'$.

We now prove the justification theorem which establishes that the above is indeed an axiomatization of special relativity:

Theorem 3. Assume $\mathsf{SpecRel}_{\mathsf{Full}}$. Then w_{mk} is a Poincaré transformation corresponding to $\mathfrak{c}$ for all inertial observers m and k.

Proof. By theorem [5, Thm 2.1, p.639] for stock SpecRel, assuming AxEField, $\mathsf{AxPh}_{\mathsf{c}}$, AxEv, AxSelf, AxSymD and that $\mathfrak{c} = 1$, imply that the worldview transformations between inertial observers are Poincaré transformations corresponding

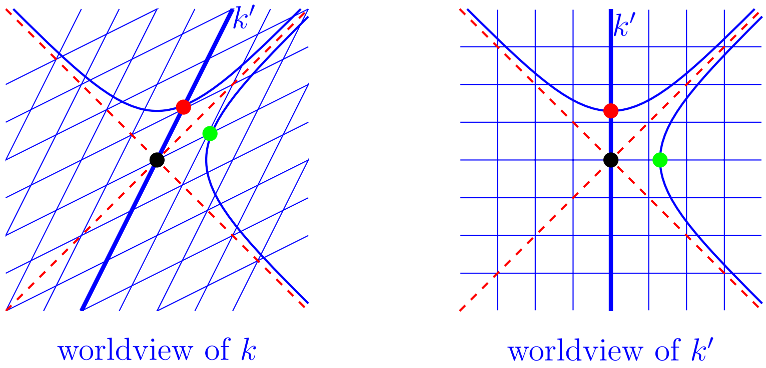

Fig. 3.11: In $\mathsf{SpecRel_{Full}}$, worldview transformations are Poincaré transformations. (In the figures, we asume that the speed of light is 1.)

to 1. Now take any model $\mathfrak{M}$ of $\mathsf{SpecRel_{Full}}$ and modify every observers coordinate systems by map

$$(t, x, y, z) \mapsto (\mathfrak{c}t, x, y, z).$$

This modified model will satisfy the required axioms of theorem [5, Thm 2.1, p.639]. Hence the worldview transformations between inertial observers in the modified model are Poincaré transformations corresponding to 1. It is straight forward to check that this means that in $\mathfrak{M}$ the worldview transformations between inertial observers were the Poincaré transformations corresponding to $\mathfrak{c}$. □

An alternative proof for Theorem 3 would be by using the Alexandrov–Zeeman[10] Theorem.

Theorem 3 shows that $\mathsf{SpecRel_{Full}}$ captures the kinematics of special relativity since it implies that the worldview transformations between inertial observers are the same as in the standard non-axiomatic approaches.

Note that the Poincaré transformations in Theorem 3 are model-dependent. When we talk about Poincaré transformations below, we mean Poincaré transformations corresponding to the speed of light of the investigated model.

Corollary 4. Assuming $\mathsf{SpecRel_{Full}}$, the speed of any inertial observer relative to any other inertial observer is slower than the speed of light:

$$(\forall h, k \in IOb)[speed_h(k) < \mathfrak{c}].$$

[10] This theorem, which states that any causal automorphism of spacetime is a Lorentz transformation up to a dilation and a translation, was independently discovered by A. D. Aleksandrov in 1949, L.-K. Hua in the 1950s and E. C. Zeeman in 1964, see [24, p.179]. See [52] for E. C. Zeeman's proof of the theorem, and [36] for a proof using definability theory.

4. USING POINCARÉ–EINSTEIN SYNCHRONISATION TO CONSTRUCT RELATIVISTIC COORDINATE SYSTEMS FOR CLASSICAL OBSERVERS

In this chapter, we are going to give a systematic translation of the formulas of $\mathsf{SpecRel}_{\mathsf{Full}}$ to the language of $\mathsf{ClassicalKin}_{\mathsf{Full}}$ such that the translation of every consequence of $\mathsf{SpecRel}_{\mathsf{Full}}$ will follow from $\mathsf{ClassicalKin}_{\mathsf{Full}}$, see Theorem 4.

The basic idea is that if classical observers use light signals and Poincaré–Einstein synchronisation, then the coordinate systems of the slower-than-light observers after a natural time adjustment will satisfy the axioms of special relativity.

Hence we will use light signals to determine simultaneity and measure distance in classical physics in the same way as those that are being used in relativity theory.

For every classical observer k, we will redefine the coordinates of events using Poincaré–Einstein synchronization. For convenience, we will work in the ether frame because there the speed of light is the same in every direction.

Let us consider the ether coordinate system e which agrees with k in every aspects possible: it intersects the worldline of observer k at the origin according to both k and e; it agrees with k on the direction of time and the directions of space axises; it agrees with k in the units of time and space.

First we consider the case where observer k is moving in the x direction according to e. We will discuss the general case when discussing Figure 4.3 below.

Let us first understand what happens in the tx-plane when classical observer k uses Poincaré–Einstein synchronization to determine the coordinates of events. Let (t_1, x_1) be an arbitrary point of the tx-plane in the coordinate system of e.

Let v be the speed of k with respect to e. Then the worldline of k according to e is defined by equation $x = vt$, as illustrated in Figure 4.1. Therefore, the worldline of the light signal sent by k in the positive x direction at instant t_0 satisfy equation $t - t_0 = \frac{x - vt_0}{\mathfrak{c}_\mathfrak{e}}$. Similarly, the light signal received by k in the negative x direction at instant t_2 satisfy equation $t - t_2 = \frac{x - vt_2}{-\mathfrak{c}_\mathfrak{e}}$. Using that (t_1, x_1) are on these lines, we get

$$\begin{aligned} \mathfrak{c}_\mathfrak{e}(t_1 - t_0) &= x_1 - vt_0, \\ -\mathfrak{c}_\mathfrak{e}(t_1 - t_2) &= x_1 - vt_2. \end{aligned} \quad . \tag{4.1}$$

Solving (4.1) for t_0 and t_2, we get

$$t_0 = \frac{\mathfrak{c}_\mathfrak{e} t_1 - x_1}{\mathfrak{c}_\mathfrak{e} - v}, \quad t_2 = \frac{\mathfrak{c}_\mathfrak{e} t_1 + x_1}{\mathfrak{c}_\mathfrak{e} + v}. \tag{4.2}$$

Let us call the coordinates of the event (t_1, x_1) in the ether coordinates and (t'_1, x'_1) in the coordinates of the moving observer. Since the light signal has the same speed in both directions, t'_1 is in the middle between t_0 and t_2, which leads to $t'_1 = \frac{t_0+t_2}{2}$ and $x'_1 = \frac{\mathfrak{c}_\mathfrak{e}(t_2-t_0)}{2}$.
Substituting (4.2) to the above equations , we find that $t'_1 = \frac{\mathfrak{c}_\mathfrak{e}^2 t_1 - v x_1}{\mathfrak{c}_\mathfrak{e}^2 - v^2}$ and $x'_1 = \frac{\mathfrak{c}_\mathfrak{e}^2 x_1 - \mathfrak{c}_\mathfrak{e}^2 v t_1}{\mathfrak{c}_\mathfrak{e}^2 - v^2}$.

$$t'_1 = \frac{\mathfrak{c}_\mathfrak{e}^2 t_1 - v x_1}{\mathfrak{c}_\mathfrak{e}^2 - v^2}, \quad x'_1 = \frac{\mathfrak{c}_\mathfrak{e}^2 x_1 - \mathfrak{c}_\mathfrak{e}^2 v t_1}{\mathfrak{c}_\mathfrak{e}^2 - v^2}.$$

Therefore, if k uses radar to coordinatize spacetime, (t_1, x_1) is mapped to $\frac{1}{\mathfrak{c}_\mathfrak{e}^2 - v^2}\left(\mathfrak{c}_\mathfrak{e}^2 t_1 - v x_1, \mathfrak{c}_\mathfrak{e}^2 x_1 - \mathfrak{c}_\mathfrak{e}^2 v t_1\right)$ when switching from the coordinate system of e to that of k.

Now, let us consider the effect of using radars to coordinatize spacetime in the directions orthogonal to the movement of k, see Figure 4.2. Consider a light signal moving with speed $\mathfrak{c}_\mathfrak{e}$ in an orthogonal direction, say the y direction, which is reflected by a mirror after traveling one time unit.

Due to the movement of the observer through the ether frame, there will be an apparent deformation of distances orthogonal to the movement: if the light signal takes one time unit relative to the ether, it will appear to only travel a distance which would, because of Pythagoras' theorem, be covered in $\sqrt{1 - v^2/\mathfrak{c}_\mathfrak{e}^2}$ time units to the observer moving along in the x direction. The same holds for the z direction. So the point with coordinates $(0, 0, 1, 1)$ in the ether coordinates will have coordinates

$$\left(0, 0, \frac{1}{\sqrt{1 - v^2/\mathfrak{c}_\mathfrak{e}^2}}, \frac{1}{\sqrt{1 - v^2/\mathfrak{c}_\mathfrak{e}^2}}\right)$$

relative to the moving observer. This explain the values on the diagonal line in the lower right corner of the Poincaré–Einstein synchronisation matrix E_v.

Consequently, the following matrix describes the Poincaré–Einstein synchronisation in classical physics:

Definition 14.

$$E_v \stackrel{\text{def}}{=} \begin{bmatrix} \frac{1}{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & \frac{-v/\mathfrak{c}_\mathfrak{e}^2}{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & 0 & 0 \\ \frac{-v}{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & \frac{1}{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} \end{bmatrix}.$$

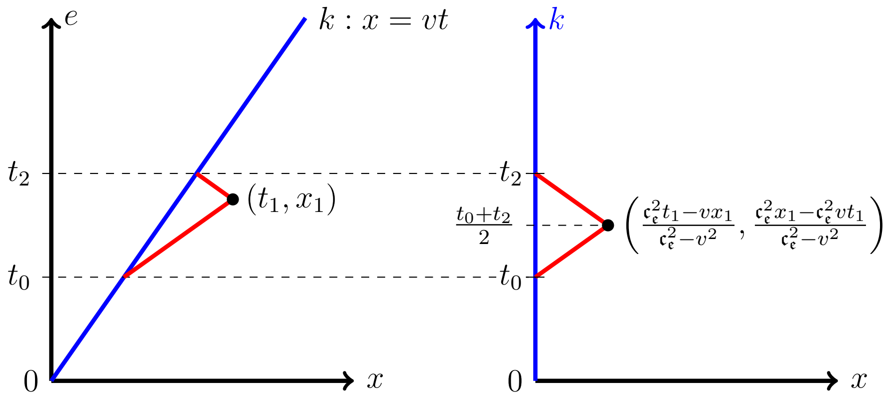

Fig. 4.1: Einstein–Poincareé synchronisation in two-dimensional classical kinematics: on the left in the coordinates system of an ether observer and on the right in the coordinate system of the moving observer. We only assume that the moving observer goes through the origin of the ether observer.

This transformation generates some asymmetry: an observer in a moving spaceship would see their spaceship shrink in the directions orthogonal to its movement relative to the coordinates of any ether observer. We can eliminate this asymmetry by multiplying with a scale factor

Definition 15. $S_v \stackrel{\text{def}}{=} \sqrt{1 - v^2/\mathfrak{c}_\mathfrak{e}^2}$

which slows the clock of k down. The combined transformation $S_v \circ E_v$ is the following Lorentz transformation:

Definition 16.

$$L_v \stackrel{\text{def}}{=} S_v \circ E_v = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & \frac{-v/\mathfrak{c}_\mathfrak{e}^2}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & 0 & 0 \\ \frac{-v}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For all $\bar{v} = (v_x, v_y, v_z) \in Q^3$ satisfying $v = |\bar{v}| < \mathfrak{c}_\mathfrak{e}$, we construct a bijection $Rad_{\bar{v}}$ (for "*radarization*") between Minkowski spacetime and Newtonian absolute spacetime.

We start by using the unique spatial rotation $R_{\bar{v}}$ that rotates $(1, v_x, v_y, v_z)$ to $(1, -v, 0, 0)$ if the ether frame is not parallel with the tx-plane:

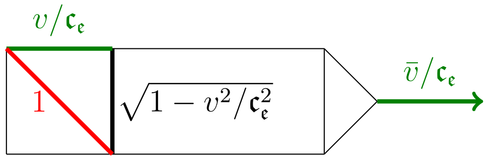

Fig. 4.2: Correction in the directions orthogonal to movement: applying Pythagoras' theorem to a spaceship moving with speed $v/\mathfrak{c}_\mathfrak{e}$.

Definition 17.

- if $v_y \neq 0$ or $v_z \neq 0$ then

$$R_{\bar{v}} \stackrel{\text{def}}{=} \frac{1}{|\bar{v}|}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -v_x & -v_y & -v_z \\ 0 & v_y & -v_x - \frac{v_z^2(|\bar{v}|-v_x)}{v_y^2+v_z^2} & \frac{v_y v_z(|\bar{v}|-v_x)}{v_y^2+v_z^2} \\ 0 & v_z & \frac{v_y v_z(|\bar{v}|-v_x)}{v_y^2+v_z^2} & -v_x - \frac{v_y^2(|\bar{v}|-v_x)}{v_y^2+v_z^2} \end{bmatrix},$$

- if $v_x \leq 0$ and $v_y = v_z = 0$ then $R_{\bar{v}}$ is the identity map,
- if $v_x > 0$ and $v_y = v_z = 0$ then

$$R_{\bar{v}} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Rotation $R_{\bar{v}}$ is only dependent on the velocity $\bar{v}$.

Then we take the Galilean boost[11] G_v that maps line $x = -vt$ to the time-axis, i.e.,

Definition 18.

$$G_v \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next we have the Lorentz transformation $S_v \circ E_v$. Finally, we use the reverse rotation $R_{\bar{v}}^{-1}$ to put the directions back to their original positions. So $Rad_{\bar{v}}$ is, as illustrated in Figure 4.3, the following composition:

[11] A Galilean boost is a time dependent translation: take a spacelike vector $\bar{v}$ and translate by $t_0\bar{v}$ in the horizontal hyperplane $t = t_0$, i.e. $(t, x, y, z) \mapsto (t, x + tv_x, y + tv_y, z + tv_z)$.

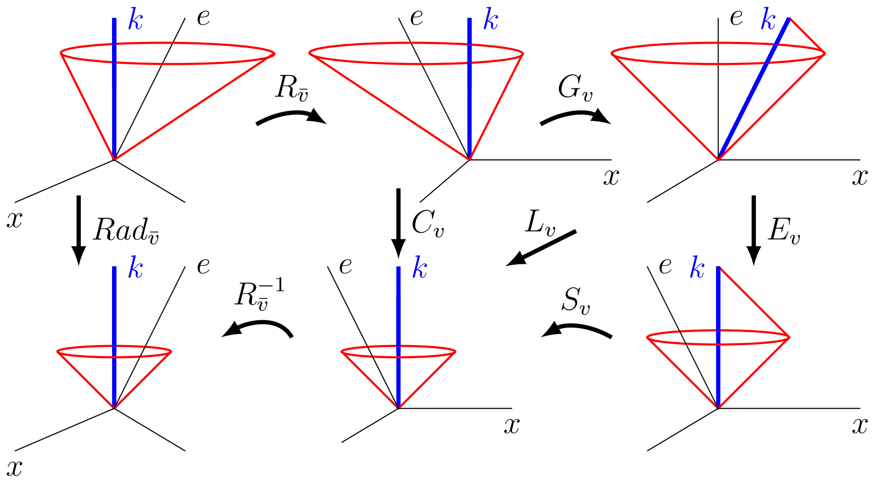

Fig. 4.3: The components of transformation $Rad_{\bar v}$: reading from the top left corner to the bottom left corner, first we put ether observer e in the kx-plane by rotation $R_{\bar v}$, we put e to the time-axis by Galilean transformation G_v, then by using Einstein-Poincaré transformation E_v we put k to the time-axis, then by scaling S_v we correct the asymmetry in the directions orthogonal to the movement, and finally we use the inverse rotation $R_{\bar v}^{-1}$ to put the direction of e back into place. The Lorentz transformation L_v and the core map C_v are also being displayed. The triangles formed by k, the outgoing lightbeam on the right of the lightcone and the incoming light beam, on the right hand size of Figure 4.3, are the same triangles as in Figure 4.1.

Definition 19.

$$Rad_{\bar v} \stackrel{\text{def}}{=} R_{\bar v}^{-1} \circ S_v \circ E_v \circ G_v \circ R_{\bar v}.$$

$Rad_{\bar v}$ is a unique well-defined linear bijection for all $|\bar v| < \mathfrak{c}_{\mathfrak{e}}$.

Definition 20. The *core map* is the transformations in between the rotations.

$$C_v \stackrel{\text{def}}{=} S_v \circ E_v \circ G_v = \begin{bmatrix} \sqrt{1 - v^2/\mathfrak{c}_{\mathfrak{e}}^2} & \frac{-v/\mathfrak{c}_{\mathfrak{e}}^2}{\sqrt{1-v^2/\mathfrak{c}_{\mathfrak{e}}^2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-v^2/\mathfrak{c}_{\mathfrak{e}}^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is worth noting that the core map is only dependent on the speed v.

We say that cone Λ is a *light cone moving with velocity* $\bar v = (v_x, v_y, v_z)$ if Λ is the translation, by a Q^4 vector, of the following cone

$$\left\{ (t, x, y, z) \in Q^4 : (x - v_x t)^2 + (y - v_y t)^2 + (z - v_z t)^2 = (\mathfrak{c}_{\mathfrak{e}} t)^2 \right\}.$$

Definition 21. We call light cones moving with velocity $(0,0,0)$ *right light cones.*

Lemma 1. Assume $\mathsf{ClassicalKin}_{\mathsf{Full}}$. Let $\bar{v} = (v_x, v_y, v_z) \in Q^3$ such that $|\bar{v}| < \mathfrak{c}_\mathfrak{e}$. Then $Rad_{\bar{v}}$ is a linear bijection that has the following properties:

1. If $\bar{v} = (0,0,0)$, then $Rad_{\bar{v}}$ is the identity map.
2. $Rad_{\bar{v}}$ maps the time-axis to the time-axis, i.e.,
$$\big(\forall \bar{y}, \bar{x} \in Q^4\big)\Big(Rad_{\bar{v}}(\bar{x}) = \bar{y} \rightarrow \big[x_1 = x_2 = x_3 = 0 \leftrightarrow y_1 = y_2 = y_3 = 0\big]\Big).$$
3. $Rad_{\bar{v}}$ scales the time-axis down by factor $\sqrt{1 - |\bar{v}|^2}$.
4. $Rad_{\bar{v}}$ transforms light cones moving with velocity $\bar{v}$ into right light cones.
5. $Rad_{\bar{v}}$ is the identity on vectors orthogonal to the plane containing the time-axis and the direction of motion of the ether frame $(0, v_x, v_y, v_z)$, i.e.,
$$\big(\forall \bar{x} \in Q^4\big)\big([x_1 v_x + x_2 v_y + x_3 v_z = 0 \wedge x_0 = 0] \rightarrow Rad_{\bar{v}}(\bar{x}) = \bar{x}\big).$$
6. The line through the origin moving with velocity $\bar{v}$ are mapped to itself and lines parallel to this line are mapped to parallel ones by $Rad_{\bar{v}}$.

Proof. To define $Rad_{\bar{v}}$, we need $\mathsf{AxEField}$ to allow us to use subtractions, divisions and square roots; AxLine because worldlines of observers must be straight lines, which also enables us to calculate speed v; and $\mathsf{AxEther}$ because we need the ether frame of reference.

$Rad_{\bar{v}}$ is well-defined because it is composed of well-defined components. Speed v is well-defined because AxLine and $\mathsf{AxEField}$.

$Rad_{\bar{v}}$ is a bijection since it is composed of bijections: Galilean transformation G is a bijection, rotations R and R^{-1} are bijections, matrix E defines a bijection, multiplication by S is a bijection (we only use the positive square root). $Rad_{\bar{v}}$ maps lines on lines since all components of $Rad_{\bar{v}}$ are linear. By definition, $Rad_{\bar{0}}$ is the identity map.

The time-axis is mapped on the time-axis by $Rad_{\bar{v}}$: $R_{\bar{v}}$ leave the time-axis in place. Galilean transformation G_v maps the time-axis to the line defined by $x = vt$. Matrix E_v maps this line back to the time-axis. S_v and $R_{\bar{v}}^{-1}$ leave the time-axis in place.

The rotations around the time axis do not change the time axis. The core map C_v scales the time axis by factor $\sqrt{1 - |\bar{v}|^2}$.

$G_v \circ R_{\bar{v}}$ transforms light cones moving with velocity $\bar{v}$ into right light cones. The rest of the transformations giving $Rad_{\bar{v}}$ map right light cones to right ones.

If $\bar{x}$ is orthogonal to $(1, 0, 0, 0)$ and $(0, \bar{v})$, then $R_{\bar{v}}$ rotates it to become orthogonal to the tx-plane. Therefore, the Galilean boost G_v and the Lorentz boost $S_v \circ E_v$ do not change the $R_{\bar{v}}$ image of $\bar{x}$. Finally $R_{\bar{v}}^{-1}$ rotates this image back to $\bar{x}$. Hence $Rad_{\bar{v}}(\bar{x}) = \bar{x}$.

The line through the origin with velocity $\bar{v} = (v_x, v_y, v_z)$, which is line e in Figure 4.3, is defined by equation system $x = v_x t$, $y = v_y t$ and $z = v_z t$. After rotation $R_{\bar{v}}$, this line has speed $-v$ in the tx plane, which core map C_v maps to itself:

$$\begin{bmatrix} \sqrt{1 - v^2/\mathfrak{c}_{\mathfrak{e}}^2} & \frac{-v/\mathfrak{c}_{\mathfrak{e}}^2}{\sqrt{1-v^2/\mathfrak{c}_{\mathfrak{e}}^2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-v^2/\mathfrak{c}_{\mathfrak{e}}^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/\mathfrak{c}_{\mathfrak{e}}^2}} \\ \frac{-v}{\sqrt{1-v^2/\mathfrak{c}_{\mathfrak{e}}^2}} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{1 - v^2/\mathfrak{c}_{\mathfrak{e}}^2}} \begin{bmatrix} 1 \\ -v \\ 0 \\ 0 \end{bmatrix}.$$

Rotation $R_{\bar{v}}^{-1}$ puts the line with speed $-v$ back on the line with velocity $\bar{v}$. Since $Rad_{\bar{v}}$ is a linear bijection, lines paralel to this line are mapped to parallel lines. $\square$

5. A FORMAL TRANSLATION OF SPECREL INTO CLASSICALKIN

In this chapter, using the radarization transformations of Chapter 4, we are give a formal translation from the language of SpecRel to that of ClassicalKin such that all the translated axioms of $\mathsf{SpecRel}_{\mathsf{Full}}$ become theorems of $\mathsf{ClassicalKin}_{\mathsf{Full}}$. To do so, we will have to translate the basic concepts of SpecRel to formulas of the language of ClassicalKin.

Since the basic concepts of the two languages use the same symbols, we indicate in a subscript whether we are speaking about the classical or the relativistic version when they are not translated identically. So we use IOb^{SR} and W^{SR} for relativistic inertial observers and worldview relation and IOb^{CK} and W^{CK} for classical inertial observers and worldview relation. Even though from the context it is always clear which language we use because formulas before the translation are in the language of SpecRel and formulas after the translation are in the language of ClassicalKin, sometimes we use this notation even in defined concepts (such as event, worldline, and worldview transformation) to help the readers.

Definition 22. $Rad_{\bar{v}_k(e)}(\bar{x})$ and its inverse $Rad^{-1}_{\bar{v}_k(e)}(\bar{y})$ are the following transformations:

$$Rad_{\bar{v}_k(e)}(\bar{x}) = \bar{y} \stackrel{\text{def}}{\iff} \big(\exists \bar{v} \in Q^3\big)[\bar{v} = \bar{v}_k(e) \land Rad_{\bar{v}}(\bar{x}) = \bar{y}]$$

$$Rad^{-1}_{\bar{v}_k(e)}(\bar{y}) = \bar{x} \stackrel{\text{def}}{\iff} \big(\exists \bar{v} \in Q^3\big)[\bar{v} = \bar{v}_k(e) \land Rad_{\bar{v}}(\bar{x}) = \bar{y}].$$

Let us now give the translation of all the basic concepts of SpecRel in the language of ClassicalKin.

Definition 23. Mathematical expressions are translated into themselves:

$$Tr(a + b = c) \stackrel{\text{def}}{\equiv} (a + b = c),$$

$$Tr(a \cdot b = c) \stackrel{\text{def}}{\equiv} (a \cdot b = c),$$

$$Tr(a < b) \stackrel{\text{def}}{\equiv} a < b.$$

Light signals are translated to light signals:

$$Tr\big(Ph(p)\big) \stackrel{\text{def}}{\equiv} Ph(p).$$

The translation of relativistic inertial observers are classical inertial observers which are slower-than-light with respect to the ether frame:

$$Tr\big(IOb^{SR}(k)\big) \stackrel{\text{def}}{\equiv} IOb^{CK}(k) \land (\forall e \in Ether)\big[speed_e(k) < \mathfrak{c}_\mathfrak{e}\big].$$

Relativistic coordinates are translated into classical coordinates by radarization[12]:

$$Tr\big(W^{SR}(k,b,\bar{x})\big) \stackrel{\text{def}}{\equiv} (\forall e \in Ether)\big[W^{CK}\big(k,b,Rad^{-1}_{\bar{v}_k(e)}(\bar{x})\big)\big].$$

Complex formulas are translated by preserving the logical connectives:

$$\begin{aligned} Tr(\neg\varphi) &\stackrel{\text{def}}{\equiv} \neg Tr(\varphi)\\ Tr(\psi \land \varphi) &\stackrel{\text{def}}{\equiv} Tr(\psi) \land Tr(\varphi)\\ Tr(\exists x[\varphi]) &\stackrel{\text{def}}{\equiv} \exists x[Tr(\varphi)]\\ &etc. \end{aligned}$$

This defines translation Tr on all formulas in the language of $\mathsf{SpecRel_{Full}}$.

Let us now see into what Tr translates the important defined concepts, such as event, worldline, worldview transformation. Worldlines are being translated as:

$$Tr\big(\bar{x} \in wl_k(b)\big) \equiv (\forall e \in Ether)\left[Rad^{-1}_{\bar{v}_k(e)}(\bar{x}) \in wl_k(b)\right]$$

and events as:

$$Tr\big(b \in ev_k(\bar{x})\big) \equiv (\forall e \in Ether)\left[b \in ev_k(Rad^{-1}_{\bar{v}_k(e)}(\bar{x}))\right].$$

Since these translations often lead to very complicated formulas, we provide some techniques to simplify translated formulas in the Appendix. In the proofs below, we will always use the simplified formulas. The simplified translation of the worldview transformation is the following:

$$Tr\big(w^{SR}_{hk}(\bar{x},\bar{y})\big) \equiv (\forall e \in Ether)\left[w^{CK}_{hk}\big(Rad^{-1}_{\bar{v}_h(e)}(\bar{x}), Rad^{-1}_{\bar{v}_k(e)}(\bar{y})\big)\right].$$

[12] By Convention 1 on page 13, a relation defined by formula $Tr\big(W^{SR}(k,b,\bar{x})\big)$ is empty if $wl_k(e)$ is not a subset of a straight line for every ether observer e (because in this case partial function $\bar{v}_k(e)$ is undefined). The same applies to the translations of the defined concepts *event*, *worldline* and *worldview transformation*.

Since $\mathsf{ClassicalKin}_{\mathsf{Full}}$ implies that w_{hk}^{CK} is a transformation (and not just a relation) if k and h are inertial observers,

$$Tr\big(w_{hk}^{SR}(\bar{x},\bar{y})\big) \equiv (\forall e \in Ether)\left[\left(Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}\right)(\bar{x}) = \bar{y}\right]$$

in this case.

Lemma 2. Assume $\mathsf{ClassicalKin}_{\mathsf{Full}}$. Then $Tr(\mathfrak{c}) \equiv \mathfrak{c}_{\mathfrak{e}}$.[13]

Proof. By $Tr(\mathsf{AxPh}_{\mathsf{c}}^{\mathsf{SR}})$ we know there is an observer-independent speed of light for the translated inertial observers. So we can choose any translated inertial observer to establish the speed of light. Ether observers are also translations of some inertial observers because they are inertial observers moving slower than $\mathfrak{c}_{\mathfrak{e}}$ with respect to ether observers. Let us take an ether observer, which in the translation has $Rad_{\bar{0}}$ being the identity. Hence, $Tr(\mathfrak{c})$ is the speed of light according to our fixed ether observer, which is $\mathfrak{c}_{\mathfrak{e}}$ in $\mathsf{ClassicalKin}_{\mathsf{Full}}$ by definition. □

Lemma 3 is helpful for proving properties of translations involving more than one observer:

Lemma 3. Assuming $\mathsf{ClassicalKin}_{\mathsf{Full}}$, if e and e' are ether observers and k and h are inertial observers slower-than-light, then

$$Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1} = Rad_{\bar{v}_k(e')} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e')}^{-1}$$

and it is a Poincaré transformation.

Proof. Let e and e' be ether observes and let k and h be inertial observers with velocities $\bar{v} = \bar{v}_k(e)$ and $\bar{u} = \bar{u}_h(e')$. By Corollary 3, $\bar{v} = \bar{u}_k(e')$ and $\bar{u} = \bar{v}_h(e)$. Therefore,

$$Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1} = Rad_{\bar{v}_k(e')} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e')}^{-1}.$$

By Theorem 2, w_{hk}^{CK} is a Galilean transformation. Trivial Galilean transformations are also (trivial) Poincaré transformations. Therefore

$$T_1 = R_{\bar{v}} \circ w_{hk}^{CK} \circ R_{\bar{u}}^{-1}$$

is a Galilean transformation because it is a composition of a Galilean transformation and two rotations, which are also (trivial) Galilean transformations. T_1

[13] That is, the translation of the defining formula of constant $\mathfrak{c}$ is equivalent to the defining formula of constant $\mathfrak{c}_{\mathfrak{e}}$ in $\mathsf{ClassicalKin}_{\mathsf{Full}}$. The same remark, with $\mathfrak{c}_{\mathfrak{e}}$ and $\mathfrak{c}$ switched and on $\mathsf{SpecRel}_{\mathsf{Full}}$, can be made for Corollary 6 below.

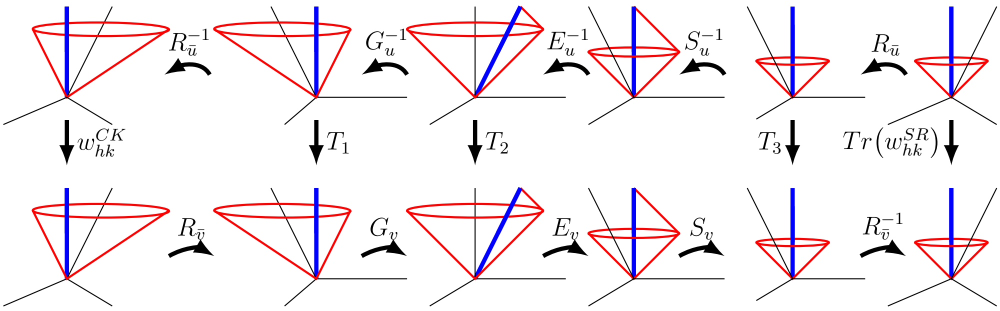

Fig. 5.1: Lemma 3: Read the figure starting in the top-right corner and follow the arrows to the left along the components of $Rad_{\bar{u}}^{-1}$, down along Galilean transformation w_{hk}^{CK} and right along the components of $Rad_{\bar{v}}$, which results in Poincaré transformation $Tr\big(w_{hk}^{SR}\big)$.

is also a trivial transformation because it is a transformation between two ether observers, which are at rest relative to each other by Corollary 1.

$$T_2 = G_v \circ T_1 \circ G_u^{-1}$$

is a (trivial) Galilean transformation. Since $S_v \circ\ E_v$ and $E_u^{-1} \circ S_u^{-1}$ are Lorentz transformations (which are special cases of Poincaré transformations),

$$T_3 = S_v \circ\ E_v \circ T_2 \circ E_u^{-1} \circ S_u^{-1}$$

is a Poincaré transformation. Since rotations are (trivial) Poincaré transformations,

$$Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1} = R_{\bar{v}}^{-1} \circ T_3 \circ R_{\bar{u}}$$

is a Poincaré transformation. □

Since, by Lemma 3, $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ leads to the same Poincaré transformation independently of the choice of ether observer e, we can use the notation $Tr\big(w_{hk}^{SR}\big)$ for this transformation, as on the right side of Figure 5.1.

Lemma 4. Assume $\mathsf{ClassicalKin_{Full}}$. Let e be an ether observer, and let k and h be inertial observers slower-than-light. Assume that w_{hk}^{CK} is a trivial transformation which is translation by vector $\bar{z}$ after linear trivial transformation T. Then $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ is the trivial transformation which is translation by vector $Rad_{\bar{v}_k(e)}(\bar{z})$ after T.

Proof. By Lemma 3, $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ is a Poincaré transformation. Since w_{hk}^{CK} is a trivial transformation, it maps vertical lines to vertical ones. By Lemma 1,

$Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ also maps vertical lines to vertical ones. Consequently, $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ is a Poincaré transformation that maps vertical lines to vertical ones. Hence it is a trivial transformation.

Let $M_{\bar{z}}$ denote the translation by vector $\bar{z}$.

By the assumptions, $w_{hk}^{CK} = M_{\bar{z}} \circ T$. The linear part T of w_{hk}^{CK} transforms the velocity of the ether frame as $(0, \bar{v}_k(e)) = T(0, \bar{v}_h(e))$ and the translation part $M_{\bar{z}}$ does not change the velocity of the ether frame. Hence $\bar{v}_k(e)$ is $\bar{v}_h(e)$ transformed by the spatial isometry part of T.

We also have that $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ is $Rad_{\bar{v}_k(e)} \circ M_{\bar{z}} \circ T \circ Rad_{\bar{v}_h(e)}^{-1}$. Since $Rad_{\bar{v}_k(e)}$ is linear, we have $Rad_{\bar{v}_k(e)} \circ M_{\bar{z}} = M_{Rad_{\bar{v}_k(e)}(\bar{z})} \circ Rad_{\bar{v}_k(e)}$. Therefore, it is enough to prove that $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1} = w_{hk}^{CK}$ if w_{hk}^{CK} is linear.

From now on, assume that w_{hk}^{CK} is linear.

Since it is a linear trivial transformation, w_{hk}^{CK} maps $(1, 0, 0, 0)$ to itself. By Item 3 of Lemma 1, $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1} = w_{hk}^{CK}$ also maps $(1, 0, 0, 0)$ to itself because $Rad_{\bar{v}_h(e)}^{-1}$ scales up the time axis the same factor as $Rad_{\bar{v}_k(e)}$ scales down because $|\bar{v}_h(e)| = |\bar{v}_k(e)|$. So $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ and w_{hk}^{CK} agree restricted to time.

Now we have to prove that $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ and w_{hk}^{CK} also agree restricted to space. By Item 5 of Lemma 1, $Rad_{\bar{v}_h(e)}^{-1}$ is identical on the vectors orthogonal to the plane containing the time-axis and the direction of motion of the ether frame, determined by vector $\bar{v}_h(e)$. Worldview transformation w_{hk}^{CK} leaves the time-axis fixed and maps the velocity of ether frame $\bar{v}_h(e)$ to $\bar{v}_k(e)$. After this $Rad_{\bar{v}_k(e)}$ does not change the vectors orthogonal to the plane containing the time-axis and the direction of motion of the ether frame. Therefore, $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ and w_{hk}^{CK} do the same thing with the vectors orthogonal to the plane containing the time-axis and the direction of motion of the ether frame, determined by vector $\bar{v}_h(e)$. So the space part of $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1}$ and w_{hk}^{CK} are isometries of Q^3 that agree on two independent vectors. This means that they are either equal or differ in a mirroring. However, they cannot differ in a mirroring as $Rad_{\bar{v}_k(e)}$ and $Rad_{\bar{v}_h(e)}^{-1}$ are orientation preserving maps because all of their components are such. Consequently, $Rad_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}^{-1} = w_{hk}^{CK}$. □

6. INTERPRETATION

Now that we have established the translation and developed the tools to simplify[14] translated formulas, we prove that it is an interpretation of $\mathsf{SpecRel}_{\mathsf{Full}}$ to $\mathsf{ClassicalKin}_{\mathsf{Full}}$.

Theorem 4. Tr is an interpretation of $\mathsf{SpecRel}_{\mathsf{Full}}$ to $\mathsf{ClassicalKin}_{\mathsf{Full}}$, i.e.,

$$\mathsf{ClassicalKin}_{\mathsf{Full}} \vdash Tr(\varphi) \quad \text{if} \quad \mathsf{SpecRel}_{\mathsf{Full}} \vdash \varphi.$$

Proof. To prove the theorem it is enough to prove that the Tr-translation of every axiom of $\mathsf{SpecRel}_{\mathsf{Full}}$ follows from $\mathsf{ClassicalKin}_{\mathsf{Full}}$. Now we will go trough all the axioms of $\mathsf{SpecRel}_{\mathsf{Full}}$ and prove their translations one by one from $\mathsf{ClassicalKin}_{\mathsf{Full}}$.

- $\mathsf{AxEField}^{\mathsf{CK}} \vdash Tr(\mathsf{AxEField}^{\mathsf{SR}})$

  Since all purely mathematical expressions are translated into themselves, $Tr(\mathsf{AxEField})$ is the axiom $\mathsf{AxEField}$ itself.

- $\mathsf{ClassicalKin}_{\mathsf{Full}} \vdash Tr(\mathsf{AxEv}^{\mathsf{SR}})$

  The translation of $\mathsf{AxEv}^{\mathsf{SR}}$ is equivalent to:

$$(\forall k,h \in IOb)\big(\forall \bar{x} \in Q^4\big)(\forall e \in Ether)\left(\begin{bmatrix} speed_e(k) < \mathfrak{c}_{\mathfrak{e}} \\ speed_e(h) < \mathfrak{c}_{\mathfrak{e}} \end{bmatrix} \rightarrow \big(\exists \bar{y} \in Q^4\big)\Big[ev_k\left(Rad^{-1}_{\bar{v}_k(e)}(\bar{x})\right) = ev_h\left(Rad^{-1}_{\bar{v}_h(e)}(\bar{y})\right)\Big]\right).$$

  To prove the formula above, let k and h be inertial observers such that $speed_e(k) < \mathfrak{c}_{\mathfrak{e}}$ and $speed_e(h) < \mathfrak{c}_{\mathfrak{e}}$ according to any Ether observer e and let $\bar{x} \in Q^4$. We have to prove that there is a $\bar{y} \in Q^4$ such that $ev_k[Rad^{-1}_{\bar{v}_k(e)}(\bar{x})] = ev_h[Rad^{-1}_{\bar{v}_h(e)}(\bar{y})]$. Let us denote $Rad^{-1}_{\bar{v}_k(e)}(\bar{x})$ by $\bar{x}'$. $\bar{x}'$ exists since $Rad_{\bar{v}_k(e)}$ is a well-defined bijection. There is a $\bar{y}'$ such that $ev_k(\bar{x}') = ev_h(\bar{y}')$ because of $\mathsf{AxEv}^{\mathsf{CK}}$. Then $\bar{y} = Rad_{\bar{v}_h(e)}(\bar{y}')$ has the required properties.

[14] See Appendix.

- $\mathsf{ClassicalKin_{Full}} \vdash Tr(\mathsf{AxSelf^{SR}})$

  The translation of $\mathsf{AxSelf^{SR}}$ is equivalent to

$$(\forall k \in IOb)(\forall e \in Ether)\Big(speed_e(k) < \mathfrak{c}_\mathfrak{e} \rightarrow (\forall \bar{y} \in Q^4)\big[W\big((k,k,Rad^{-1}_{\bar{v}_k(e)}(\bar{y})\big) \leftrightarrow y_1 = y_2 = y_3 = 0\big]\Big).$$

  To prove the formula above, let k be an inertial observer such that $speed_e(k) < \mathfrak{c}_\mathfrak{e}$ according to any Ether observer e and let $\bar{y} \in Q^4$. We have to prove that $W\big((k,k,Rad^{-1}_{\bar{v}_k(e)}(\bar{y})\big)$ if and only if $y_1 = y_2 = y_3 = 0$. Let $\bar{x} \in Q^4$ be such that $Rad_{\bar{v}_k(e)}(\bar{x}) = \bar{y}$. By $\mathsf{AxSelf^{CK}}$, $W\big((k,k,\bar{x})\big)$ if and only if $x_1 = x_2 = x_3 = 0$. This holds if and only if $y_1 = y_2 = y_3 = 0$ since by item 2 of Lemma 1 $Rad_{\bar{v}}$ transformation maps the time-axis on the time-axis.

- $\mathsf{ClassicalKin_{Full}} \vdash Tr(\mathsf{AxSymD^{SR}})$

  The translation of $\mathsf{AxSymD^{SR}}$ is equivalent to:

$$(\forall k,k' \in IOb)\big(\forall \bar{x},\bar{y},\bar{x}',\bar{y}' \in Q^4\big)(\forall e \in Ether)\left(\begin{bmatrix} speed_e(k) < \mathfrak{c}_\mathfrak{e} \wedge speed_e(k') < \mathfrak{c}_\mathfrak{e}\big) \\ time(\bar{x},\bar{y}) = time(\bar{x}',\bar{y}') = 0 \\ ev_k(Rad^{-1}_{\bar{v}_k(e)}(\bar{x})) = ev_{k'}(Rad^{-1}_{\bar{v}_{k'}(e)}(\bar{x}')) \\ ev_k(Rad^{-1}_{\bar{v}_k(e)}(\bar{y})) = ev_{k'}(Rad^{-1}_{\bar{v}_{k'}(e)}(\bar{y}')) \end{bmatrix} \rightarrow space(\bar{x},\bar{y}) = space(\bar{x}',\bar{y}')\right).$$

  Let k and k' be inertial observers, let $\bar{x}$, $\bar{y}$, $\bar{x}'$, and $\bar{y}'$ be coordinate points, and let e be an ether observer such that $speed_e(k) < \mathfrak{c}_\mathfrak{e}$, $speed_e(k') < \mathfrak{c}_\mathfrak{e}$, $time(\bar{x},\bar{y}) = time(\bar{x}',\bar{y}') = 0$, $ev_k(Rad^{-1}_{\bar{v}_k(e)}(\bar{x})) = ev_{k'}(Rad^{-1}_{\bar{v}_{k'}(e)}(\bar{x}'))$, and $ev_k(Rad^{-1}_{\bar{v}_k(e)}(\bar{y})) = ev_{k'}(Rad^{-1}_{\bar{v}_{k'}(e)}(\bar{y}'))$. Let $P = Rad_{v_{k'}(e)} \circ w_{kk'} \circ Rad^{-1}_{v_k(e)}$. By Lemma 3, P is a Poincaré transformation. By the assumptions, $P(\bar{x}) = \bar{x}'$ and $P(\bar{y}) = \bar{y}'$. Therefore, $time(\bar{x},\bar{y})^2 - space(\bar{x},\bar{y})^2 = time(\bar{x}',\bar{y}')^2 - space(\bar{x}',\bar{y}')^2$. Since both $time(\bar{x},\bar{y})$ and $time(\bar{x}',\bar{y}')$ are zero, $space(\bar{x},\bar{y})^2 = space(\bar{x}',\bar{y}')^2$. Consequently, $space(\bar{x},\bar{y}) = space(\bar{x}',\bar{y}')$ because they are both positive quantities.

- $\mathsf{ClassicalKin_{Full}} \vdash Tr(\mathsf{AxLine^{SR}})$

The translation of $\mathsf{Line}^{\mathsf{SR}}$ is equivalent to:

$$(\forall k, h \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{z} \in Q^4\big)\big(\forall e \in Ether\big)\Bigg[\begin{bmatrix} speed_e(k) < \mathfrak{c}_{\mathfrak{e}} \\ speed_e(h) < \mathfrak{c}_{\mathfrak{e}} \\ Rad^{-1}_{\bar{v}_k(e)}(\bar{x}), Rad^{-1}_{\bar{v}_k(e)}(\bar{y}), Rad^{-1}_{\bar{v}_k(e)}(\bar{z}) \in wl_k(h) \end{bmatrix} \rightarrow (\exists a \in Q)\big[\bar{z} - \bar{x} = a(\bar{y} - \bar{x}) \vee \bar{y} - \bar{z} = a(\bar{z} - \bar{x})\big]\Bigg].$$

Because of $\mathsf{AxLine}^{\mathsf{CK}}$, $Rad^{-1}_{\bar{v}_k(e)}(\bar{x})$, $Rad^{-1}_{\bar{v}_k(e)}(\bar{y})$ and $Rad^{-1}_{\bar{v}_k(e)}(\bar{z})$ are on a straight line. Since $Rad_{\bar{v}}$ is a linear map, $\bar{x}$, $\bar{y}$ and $\bar{z}$ are on a straight line, hence the translation of $\mathsf{AxLine}^{\mathsf{SR}}$ follows.

- $\mathsf{ClassicalKin}_{\mathsf{Full}} \vdash Tr(\mathsf{AxTriv}^{\mathsf{SR}})$

  The translation of $\mathsf{AxTriv}^{\mathsf{SR}}$ is equivalent to:

$$(\forall T \in Triv)(\forall h \in IOb)(\forall e \in Ether)\left(speed_e(h) < \mathfrak{c}_{\mathfrak{e}} \rightarrow (\exists k \in IOb)\begin{bmatrix} speed_e(k) < \mathfrak{c}_{\mathfrak{e}} \\ Rad_{\bar{v}_k(e)} \circ w_{hk} \circ Rad^{-1}_{\bar{v}_h(e)} = T \end{bmatrix}\right).$$

  To prove $Tr(\mathsf{AxTriv}^{\mathsf{SR}})$, we have to find a slower-than-light inertial observer k for every trivial transformation T and a slower-than-light inertial observer h such that $Rad_{\bar{v}_k(e)} \circ w_{hk} \circ Rad^{-1}_{\bar{v}_h(e)} = T$.

  By $\mathsf{AxTriv}^{\mathsf{CK}}$ and Lemma 4, there is an inertial observer k such that $Rad_{\bar{v}_k(e)} \circ w^{CK}_{hk} \circ Rad^{-1}_{\bar{v}_h(e)} = T$. This k is also slower-than-light since w^{CK}_{hk} is a trivial transformation.

- $\mathsf{ClassicalKin}_{\mathsf{Full}} \vdash Tr(\mathsf{AxPh}^{\mathsf{SR}}_{\mathsf{c}})$

  The translation of $\mathsf{AxPh}^{\mathsf{SR}}_{\mathsf{c}}$ is equivalent to:

$$(\exists c \in Q)\Bigg(c > 0 \wedge (\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big)(\forall e \in Ether)\Bigg[speed_e(k) < \mathfrak{c}_{\mathfrak{e}} \rightarrow (\exists p \in Ph)\left(\begin{bmatrix} W^{CK}\big(k, p, Rad^{-1}_{\bar{v}_k(e)}(\bar{x})\big) \\ W^{CK}\big(k, p, Rad^{-1}_{\bar{v}_k(e)}(\bar{y})\big) \end{bmatrix} \leftrightarrow space(\bar{x}, \bar{y}) = c \cdot time(\bar{x}, \bar{y})\right)\Bigg]\Bigg).$$

It is enough to show that any slower-than-light inertial observers k can send a light signal trough coordinate points $\bar{x}'$ and $\bar{y}'$ exactly if

$$space\big(Rad_{\bar{v}_k(e)}(\bar{x}'), Rad_{\bar{v}_k(e)}(\bar{y}')\big) = \mathfrak{c}_{\mathfrak{e}} \cdot time\big(Rad_{\bar{v}_k(e)}(\bar{x}'), Rad_{\bar{v}_k(e)}(\bar{y}')\big)$$

holds for any ether observer e, i.e., if $Rad_{\bar{v}_k(e)}(\bar{x}')$ and $Rad_{\bar{v}_k(e)}(\bar{y}')$ are on a right light cone. k can send a light signal through coordinate points $\bar{x}'$ and $\bar{y}'$ if they are on a light cone moving with velocity $\bar{v}_k(e)$ by Theorem 2 because Galilean transformation w_{ek} maps worldlines of light signals to worldlines of light signals and light signals move along right light cones according to e by AxEther. $Rad_{\bar{v}_k(e)}$ transform these cones into right light cones by Item 4 of Lemma 1. Therefore, $Rad_{\bar{v}_k(e)}(\bar{x}')$ and $Rad_{\bar{v}_k(e)}(\bar{y}')$ are on a right light cone if k can send a light signal trough $\bar{x}'$ and $\bar{y}'$, and this is what we wanted to show.

- $\mathsf{ClassicalKin}_{\mathsf{Full}} \vdash Tr(\mathsf{AxThExp}^{\mathsf{SR}})$

  The translation of $\mathsf{AxThExp}^{\mathsf{SR}}$ is equivalent to:

$$(\exists h \in IOb)(\forall e \in Ether)\big[speed_e(h) < \mathfrak{c}_{\mathfrak{e}}\big] \\ \wedge (\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big)(\forall e \in Ether) \\ \left(\begin{bmatrix} speed_e(k) < \mathfrak{c}_{\mathfrak{e}} \\ space(\bar{x}, \bar{y}) < \mathfrak{c}_{\mathfrak{e}} \cdot time(\bar{x}, \bar{y}) \end{bmatrix} \rightarrow (\exists k' \in IOb) \begin{bmatrix} speed_e(k') < \mathfrak{c}_{\mathfrak{e}} \\ x, y \in wl_k(k') \end{bmatrix} \right).$$

The first conjunct of the translation follows immediately from AxEther. Let us now prove the second conjunct. From $\mathsf{AxThExp}_+$ we get inertial observers both inside and outside of the light cones. Those observers which are inside the light cone (which are the ones we are interested in) stay inside the light cone by the translation by Items 2 and 4 of Lemma 1. Since we have only used the fact that there are observers on every straight line inside the light cones, this proof of $Tr(\mathsf{AxThExp}^{\mathsf{SR}})$ goes trough also for the $NoFTL$ case, needed in Theorem 6 below. □

Theorem 5. There is no interpretation of $\mathsf{ClassicalKin}_{\mathsf{Full}}$ to $\mathsf{SpecRel}_{\mathsf{Full}}$.

Proof. Assume for the purpose of a reductio argument that Tr' is an interpretation of $\mathsf{ClassicalKin}_{\mathsf{Full}}$ to $\mathsf{SpecRel}_{\mathsf{Full}}$. Then Tr' would turn every model of $\mathsf{SpecRel}_{\mathsf{Full}}$ into a model of $\mathsf{ClassicalKin}_{\mathsf{Full}}$. There is a model $\mathfrak{M}$ of $\mathsf{SpecRel}_{\mathsf{Full}}$ such that $B_{\mathfrak{M}} = IOb_{\mathfrak{M}} \cup Ph_{\mathfrak{M}}$ and the automorphism group of $\mathfrak{M}$ acts transitively on both $IOb_{\mathfrak{M}}$ and $Ph_{\mathfrak{M}}$, see, e.g., the second model constructed in [46, Thm.2]. That is, for any two inertial observers k and h in $\mathfrak{M}$, there is an automorphism of $\mathfrak{M}$ taking k to h, and the same is true for any two light signals in $\mathfrak{M}$.

Let $\mathfrak{M}'$ be the model of $\mathsf{ClassicalKin}_{\mathsf{Full}}$ that $\mathfrak{M}$ is turned into by translating the basic concepts of $\mathsf{ClassicalKin}_{\mathsf{Full}}$ to defined concepts of $\mathsf{SpecRel}_{\mathsf{Full}}$ via Tr'. Then every automorphism of $\mathfrak{M}$ is also an automorphism of $\mathfrak{M}'$. Since Tr' has to translate bodies into bodies, there have to be two sets of bodies in $\mathfrak{M}'$ such that any two bodies from the same set can be mapped to each other by an automorphism of $\mathfrak{M}'$. However, in models of $\mathsf{ClassicalKin}_{\mathsf{Full}}$, observers moving with different speeds relative to the ether frame cannot be mapped to each other by an automorphism. By $\mathsf{AxThExp}_{+}$ there are inertial observers in $\mathfrak{M}$ moving with every finite speed. Therefore, there are infinitely many sets of inertial observers which elements cannot be mapped into each other by an automorphism. This is a contradiction showing that no such model $\mathfrak{M}'$ and hence no such translation Tr' can exist. □

Corollary 5. There is no definitional equivalence between $\mathsf{SpecRel}_{\mathsf{Full}}$ and $\mathsf{ClassicalKin}_{\mathsf{Full}}$.

7. INTERMEZZO: THE MICHELSON–MORLEY EXPERIMENT

As an illustration, we will now show how the null result of the Michelson–Morley experiment [32] behaves under our interpretation. This is a complicated experiment involving interferometry to measure the relative speed between two light signals, but we can make abstraction of that here.

Let, as illustrated on the right side of Figure 7.1, inertial observer k send out two light signals at time $-L$, perpendicular to each other in the x and y direction. Let us assume that observer k sees the ether frame moving with speed $-v$ in the tx-plane — such that we do not have to rotate into the tx-plane (i.e. $\bar{v} = (-v, 0, 0)$ so that $R_{\bar{v}}$ is the identity as defined in Definition 17). The light signals are reflected by mirrors at $(0, L, 0, 0)$ and $(0, 0, L, 0)$, assuming that the speed of light is 1. In accordance with the null result of the Michelson–Morley experiment, the reflected light signals are both received by observer k at time L.

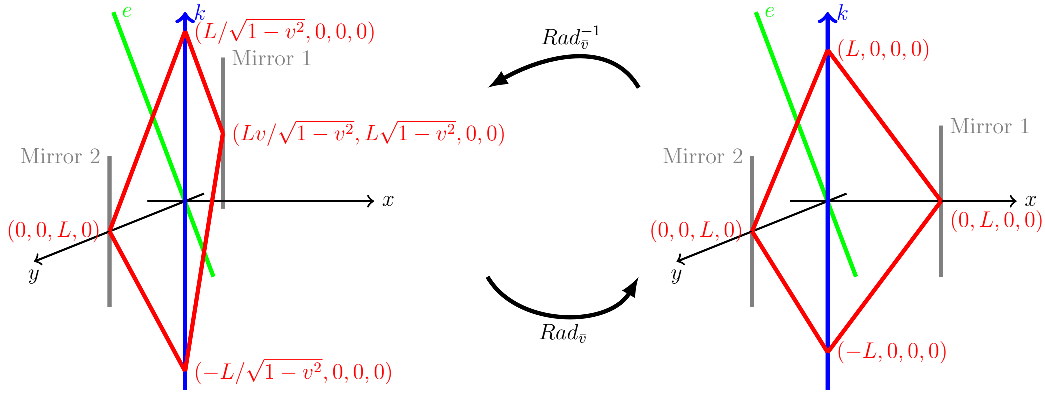

Fig. 7.1: On the left we have the classical setup which is transformed by $Rad_{\bar{v}}$ into the setup on the right for the Michelson–Morley experiment. Both setups are part of models of classical kinematics. As in all illustrations, we have chosen our units such that the speed of light is 1.

To understand how this typical relativistic setup can be modeled in classical kinematics, we have to find which classical setup is being transformed into it. So we need the inverse of the radarisation $Rad_{\bar{v}}^{-1}$, which since the ether is in the tx-plane is just the inverse of the core map C_v^{-1}:

$$Rad_{\bar{v}}^{-1} = C_v^{-1} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & \frac{v/\mathfrak{c}_\mathfrak{e}^2}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & 0 & 0 \\ 0 & \sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We use this inverse core map to find which coordinates are being transformed into the coordinates on the right side of figure 7.1:

$$\begin{bmatrix} \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & \frac{v/\mathfrak{c}_\mathfrak{e}^2}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & 0 & 0 \\ 0 & \sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} L \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{L}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & \frac{v/\mathfrak{c}_\mathfrak{e}^2}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & 0 & 0 \\ 0 & \sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -L \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{L}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & \frac{v/\mathfrak{c}_\mathfrak{e}^2}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & 0 & 0 \\ 0 & \sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ L \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{Lv/\mathfrak{c}_\mathfrak{e}^2}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} \\ L\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2} \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & \frac{v/\mathfrak{c}_\mathfrak{e}^2}{\sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2}} & 0 & 0 \\ 0 & \sqrt{1-v^2/\mathfrak{c}_\mathfrak{e}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ L \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L \\ 0 \end{bmatrix}.$$

Using these coordinates, we can draw the left side of the figure, which is the classical setup which is translated by $Rad_{\bar{v}}$ into our setup of the Michelson–Morley experiment. On the left side of the figure, we see that the speed of light is not the same in every direction according to observer k. This illustrates that our translation does not preserve simultaneity (the time at which the light beams hit the mirrors are not the same anymore), speed, distance and time difference.

Note that while our translation function Tr translates axioms of special relativity theory into theorems of classical kinematics, models are transformed the other way round from classical mechanics to special relativity theory.

8. DEFINITIONAL EQUIVALENCE

We will now slightly modify our axiom systems $\mathsf{SpecRel_{Full}}$ and $\mathsf{ClassicalKin_{Full}}$ to establish a definitional equivalence between them. This will provide us with an insight about the exact differences between special relativity and classical kinematics.

Definition 24. An *observer* is any body which can coordinatize other bodies:

$$Ob(k) \stackrel{\text{def}}{\iff} (\exists b \in B)\big(\exists \bar{x} \in Q^4\big)\big(W(k, b, \bar{x})\big).$$

Since we will be translating back and forth, we need a guarantee that all observers have a translation.

<u>$\mathsf{AxNoAcc}$</u> All observers are inertial observers:

$$(\forall k \in B)[W(Ob(k) \to IOb(k)].$$

Until now, the existence of non-inertial observers, such as accellerating observers, was not forbidden: translation Tr just ignored them, without any effect on the interpretation. However, for definitional equivalence, we need to be certain that there exists a bijection between the classes of all models of both theories.

To make classical kinematics equivalent to special relativity, we have to ban the inertial observers that are not move slower-than-light relative to the ether frame. This is done by the next axiom.

<u>$\mathsf{AxNoFTL}$</u> All inertial observers move slower-than-light with respect to the ether frames:

$$(\forall k \in IOb)(\forall e \in Ether)\big[speed_e(k) < \mathfrak{c}_\mathfrak{e}\big].$$

Axiom $\mathsf{AxNoFTL}$ contradicts $\mathsf{AxThExp_+}$. Therefore, we replace $\mathsf{AxThExp_+}$ with the following weaker assumption.

<u>$\mathsf{AxThExp^{STL}}$</u> Inertial observers can move with any speed which is in the ether frame slower than that of light:

$$(\exists h \in B)[IOb(h)] \land (\forall e \in Ether)\big(\forall \bar{x}, \bar{y} \in Q^4\big)\big(space(\bar{x}, \bar{y}) < \mathfrak{c}_\mathfrak{e} \cdot time(\bar{x}, \bar{y}) \to (\exists k \in IOb)\big[\bar{x}, \bar{y} \in wl_e(k)\big]\big).$$

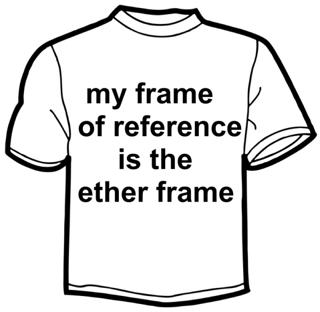

Fig. 8.1: "Just pick any relativistic observer and give him a t-shirt. Then we call him, and all observers who are at rest relative to him, *primitive ether observer.*" This can be done because all relativistic observers agree on the speed of light.

Let us now introduce our axiom system $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ as follows:

$$\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \stackrel{\text{def}}{=} \mathsf{ClassicalKin}_{\mathsf{Full}} \cup \{\mathsf{AxNoAcc}, \mathsf{AxNoFTL}, \mathsf{AxThExp}^{\mathsf{STL}}\} \setminus \{\mathsf{AxThExp}_+\}.$$

To make special relativity equivalent to classical kinematics, we have to introduce a class of observers which will play the role of Ether, for which we have to extend our language with a unary relation E to

$$\{\, B, Q\,; IOb, Ph, E, +, \cdot, \leq, W \,\}.$$

We call the set defined by E the *primitive ether frame*, and its elements *primitive ether observers.* They are *primitive* in the sense that they are concepts which are solely introduced by an axiom without definition:

<u>$\mathsf{AxPrimitiveEther}$</u> There is a non-empty class of distinguished observers, stationary with respect to each other, which is closed under trivial transformations:

$$(\exists e \in IOb)\Big[(\forall k \in B)\Big(\begin{bmatrix} IOb(k) \\ (\exists T \in Triv) w^{SR}_{ek} = T \end{bmatrix} \leftrightarrow E(k)\Big)\Big].$$

Let us now introduce our axiom system $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ as follows:

$$\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \stackrel{\text{def}}{=} \mathsf{SpecRel}_{\mathsf{Full}} \cup \{\mathsf{AxNoAcc}, \mathsf{AxPrimitiveEther}\}.$$

Definition 25. Translation Tr_+ from $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ to $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ is the translation that restricted to $\mathsf{SpecRel}_{\mathsf{Full}}$ is Tr:

$$Tr_+|_{\mathsf{SpecRel}_{\mathsf{Full}}} \stackrel{\text{def}}{=} Tr,$$

while the translation of the primitive ether is the classical ether:

$$Tr_+\big(E(x)\big) \stackrel{\text{def}}{\equiv} Ether(x).$$

Definition 26. The inverse translation Tr'_+ from $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ to $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ is the translation that translates the basic concepts as:

$$Tr'_+(a+b=c) \stackrel{\text{def}}{\equiv} (a+b=c),$$

$$Tr'_+(a\cdot b=c) \stackrel{\text{def}}{\equiv} (a\cdot b=c),$$

$$Tr'_+(a<b) \stackrel{\text{def}}{\equiv} a<b,$$

$$Tr'_+\big(Ph^{CK}(p)\big) \stackrel{\text{def}}{\equiv} Ph^{SR}(p),$$

$$Tr'_+\big(IOb^{CK}(b)\big) \stackrel{\text{def}}{\equiv} IOb^{SR}(b),$$

$$Tr'_+\big(W^{CK}(k,b,\bar{x})\big) \stackrel{\text{def}}{\equiv} (\forall e\in E)\big[W^{SR}\big(k,b,Rad_{\bar{v}_k(e)}(\bar{x})\big)\big].$$

By Theorem 8, the two slight modifications above are enough to make classical kinematics and special relativity equivalent.

Lemma 5. Assume $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$. Then $Tr'_+[Ether(b)] \equiv E(b)$.

Proof. By the definition of $Ether$, $Tr'_+[Ether(b)]$ is equivalent to

$$Tr'_+\left(\begin{array}{c} IOb(b)\wedge(\exists c\in Q)\Big[c>0\wedge\big(\forall\bar{x},\bar{y}\in Q^4\big) \\ \Big((\exists p\in Ph)\big(\big[\bar{x},\bar{y}\in wl_b(p)\big]\leftrightarrow space(\bar{x},\bar{y})=c\cdot time(\bar{x},\bar{y})\big)\Big)\Big]\end{array}\right),$$

which is by defintion of Tr'_+ equivalent to

$$\begin{aligned} &IOb(b)\wedge(\exists c\in Q)\Big[c>0\wedge\big(\forall\bar{x},\bar{y}\in Q^4\big)\\ &\qquad\Big((\exists p\in Ph)\big(Tr'_+\big[[\bar{x},\bar{y}\in wl_b(p)]\big]\leftrightarrow space(\bar{x},\bar{y})=c\cdot time(\bar{x},\bar{y})\big)\Big)\Big],\end{aligned}$$

which by definition of worldlines is equivalent to

$$\begin{aligned} &IOb(b)\wedge(\exists c\in Q)\Big[c>0\wedge\big(\forall\bar{x},\bar{y}\in Q^4\big)\\ &\quad\Big((\exists p\in Ph)\big(Tr'_+\big[\,W(b,p,\bar{x})\wedge W(b,p,\bar{y})\big]\leftrightarrow space(\bar{x},\bar{y})=c\cdot time(\bar{x},\bar{y})\big)\Big)\Big],\end{aligned}$$

which by definition of $Tr'_+(W)$ is equivalent to

$$\begin{aligned} &IOb(b)\wedge(\exists c\in Q)\Big[c>0\wedge\big(\forall\bar{x},\bar{y}\in Q^4\big)\\ &\qquad\qquad\Big((\exists p\in Ph)\big(\big[(\forall e\in E)\big(W^{SR}[b,p,Rad_{\bar{v}_b(e)}(\bar{x})]\big)\\ &\qquad\wedge(\forall e\in E)\big(W^{SR}[b,p,Rad_{\bar{v}_b(e)}(\bar{y})]\big)\big]\leftrightarrow space(\bar{x},\bar{y})=c\cdot time(\bar{x},\bar{y})\big)\Big)\Big],\end{aligned}$$

which by definition of worldlines is equivalent to

$$IOb(b) \wedge (\exists c \in Q)\Big[c > 0 \wedge \big(\forall \bar{x}, \bar{y} \in Q^4\big)\Big(\big(\exists p \in Ph\big)\big((\forall e \in E)[Rad_{\bar{v}_b(e)}(\bar{x}), Rad_{\bar{v}_b(e)}(\bar{y}) \in wl_b(p)] \leftrightarrow space(\bar{x}, \bar{y}) = c \cdot time(\bar{x}, \bar{y})\big)\Big)\Big].$$

This means that b is an inertial observer which after transforming its worldview by $Rad_{\bar{v}(e)}$ sees the light signals moving along right light cones. Since the light cones are already right ones in $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ and $Rad_{\bar{v}_b(e)}$ would ruin light cones if $\bar{v}_b(e) \neq (0, 0, 0)$ because of Item 1 of Lemma 1 and because $Rad_{\bar{v}_b(e)}$ is a bijection, b must be stationary relatively to primitive ether e, and hence the above is equivalent to $E(b)$. □

Corollary 6. Assume $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$. Then $Tr'_+(\mathfrak{c}_\mathfrak{e}) \equiv \mathfrak{c}$.

Proof. Since the ether is being translated into the primitive ether, the speed of light $\mathfrak{c}_\mathfrak{e}$ in the ether frame is translated to the speed of light of the primitive ether. In $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ the speed of light $\mathfrak{c}$ is the same for all observers. □

Theorem 6. Tr_+ is an interpretation of $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ to $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$, i.e.,

$$\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr_+(\varphi) \quad \text{if} \quad \mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash \varphi.$$

Proof. The proof of this theorem is basically the same as that of Theorem 4. The only differences are the proof of $Tr_+(\mathsf{AxThExp})$ since $\mathsf{AxThExp}_+$ is replaced by $\mathsf{AxThExp}^{\mathsf{STL}}$, and that we also need to prove $Tr_+(\mathsf{AxPrimitiveEther})$ and $Tr_+(\mathsf{AxNoAcc})$.

- The proof for $Tr(\mathsf{AxThExp})$ goes through since there we were only using that there are observers on every straight line inside of the light cones, which is covered by $\mathsf{AxThExp}^{\mathsf{STL}}$.
- $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr_+(\mathsf{AxPrimitiveEther})$.

  The translation of $\mathsf{AxPrimitiveEther}$ is

  $$Tr_+\Big[(\exists e \in IOb)\big[(\forall k \in B)\big([IOb(k) \wedge (\exists T \in Triv)w^{SR}_{ek} = T] \leftrightarrow E(k)\big)\big]\Big],$$

  which by the previously established translations of IOb, w and $Ether$, is equivalent to

  $$(\exists e \in IOb)\Big((\forall e' \in Ether)[speed_{e'}(e) < \mathfrak{c}_\mathfrak{e}] \wedge \Big[(\forall k \in B)\big(\big[IOb(k) \wedge (\forall e' \in Ether)[speed_{e'}(k) < \mathfrak{c}_\mathfrak{e}] \wedge (\exists T \in Triv)(\forall e' \in Ether)\big(\forall \bar{x}, \bar{y} \in Q^4\big) w^{CK}_{ek}(Rad^{-1}_{\bar{v}_e(e')}(\bar{x}), Rad^{-1}_{\bar{v}_k(e')}(\bar{y})) = T(\bar{x}, \bar{y})\big] \leftrightarrow Ether(k)\big)\Big]\Big),$$

which, using the result of the Appendix, can be simplified to

$$(\exists e \in IOb)(\forall e' \in Ether)\Big[(speed_{e'}(e) < \mathfrak{c}_{\mathfrak{e}})\wedge$$
$$\Big((\forall k \in B)\big[\big(IOb(k) \wedge [speed_{e'}(k) < \mathfrak{c}_{\mathfrak{e}}] \wedge (\exists T \in Triv)\big(\forall \bar{x}, \bar{y} \in Q^4\big)$$
$$w_{ek}^{CK}(Rad^{-1}_{\bar{v}_e(e')}(\bar{x}), Rad^{-1}_{\bar{v}_k(e')}(\bar{y})) = T(\bar{x},\bar{y})\big) \leftrightarrow Ether(k)\big]\Big)\Big].$$

The $w_{ek}^{CK}(Rad^{-1}_{\bar{v}_e(e')}(\bar{x}), Rad^{-1}_{\bar{v}_k(e')}(\bar{y})) = T(\bar{x},\bar{y})$ part in the above translation can be written as $Rad_{\bar{v}_k(e')} \circ w_{ek}^{CK} \circ Rad^{-1}_{\bar{v}_e(e')} = T$, from which, by Item 5 of Lemma 1, follows that there exists a trivial transformation T' such that $w_{ek}^{CK} = Rad^{-1}_{\bar{v}_k(e')} \circ T \circ Rad_{\bar{v}_e(e')} = T'$. So Tr_+(AxPrimitiveEther) is equivalent to

$$(\exists e \in IOb)(\forall e' \in Ether)\Big[[speed_{e'}(e) < \mathfrak{c}_{\mathfrak{e}}]\wedge$$
$$\big((\forall k \in B)\big[\big(IOb(k) \wedge [speed_{e'}(k) < \mathfrak{c}_{\mathfrak{e}}]\wedge$$
$$(\exists T' \in Triv)\big(\forall \bar{x}, \bar{y} \in Q^4\big) w_{ek}^{CK}(\bar{x},\bar{y}) = T'(\bar{x},\bar{y})\big) \leftrightarrow Ether(k)\big]\big)\Big].$$

To prove this from $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$, let e be an ether observer. Then $IOb(e)$ holds and $speed_{e'}(e) = 0 < \mathfrak{c}_{\mathfrak{e}}$ for every $e' \in Ether$. Therefore, we only have to prove that k is an ether observer if and only if it is a slower-than-light inertial observer such that w_{ek}^{CK} is some trivial transformation.

If k is an ether observer, then k is a slower-than-light inertial observer and w_{ek}^{CK} is indeed a trivial transformation because it is a Galilean transformation between two ether observers which are stationary relative to each other. The other direction of the proof is that if k is an inertial observer which is transformed into an ether observer by a trivial transformation, then k is itself an ether observer because then k also sees the light cones right.

- $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr_+(\mathsf{AxNoAcc})$.

  The translation of AxNoAcc is equivalent to:

$$(\forall k \in B)(\exists \bar{x} \in Q^4)(\exists b \in B)(\forall e \in Ether)$$
$$\big[W^{CK}\big(k, b, Rad^{-1}_{\bar{v}_k(e)}(\bar{x})\big) \rightarrow \big(IOb^{CK}(k) \wedge [speed_e(k) < \mathfrak{c}_{\mathfrak{e}}]\big)\big].$$

  which follows directly from AxNoAcc since $Rad_{\bar{v}_k(e)}$ is the same bijection for all ether observers e, and from AxNoFTL. □

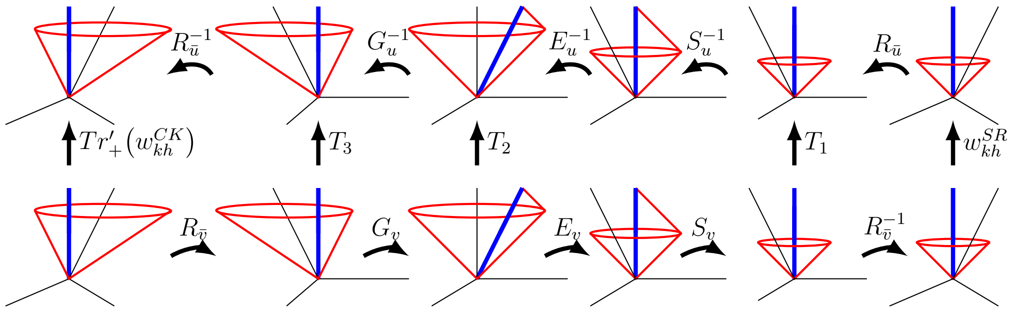

Fig. 8.2: Lemma 6: Read the figure starting in the bottom-left corner and follow the arrows to the right along the components of $Rad_{\bar{v}}$, up along Poincaré transformation w_{kh}^{SR} and left along the components of $Rad_{\bar{u}}^{-1}$, which results in Galilean transformation $Tr'_+(w_{kh}^{CK})$.

Lemma 6. Assuming $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$, if e and e' are primitive ether observers and k and h are inertial observers, then

$$Rad^{-1}_{\bar{v}_h(e)} \circ w^{SR}_{kh} \circ Rad_{\bar{v}_k(e)} = Rad^{-1}_{\bar{v}_h(e')} \circ w^{SR}_{kh} \circ Rad_{\bar{v}_k(e')}$$

and it is a Galilean transformation.

Proof. Let e and e' be primitive ether observes and let k and h be inertial observers with velocities $\bar{v} = \bar{v}_k(e)$ and $\bar{u} = \bar{v}_h(e')$. By the definition of primitive ether, all primitive ethers are at rest relative to each other, from which follows that $\bar{v} = \bar{u}_k(e')$ and $\bar{u} = \bar{v}_h(e)$. Therefore,

$$Rad^{-1}_{\bar{v}_h(e)} \circ w^{SR}_{kh} \circ Rad_{\bar{v}_k(e)} = Rad^{-1}_{\bar{v}_h(e')} \circ w^{SR}_{kh} \circ Rad_{\bar{v}_k(e')}.$$

By Theorem 3, w^{SR}_{kh} is a Poincaré transformation. Therefore

$$T_1 = R_{\bar{u}} \circ w^{SR}_{kh} \circ R^{-1}_{\bar{v}}$$

is a Poincaré transformation because it is a composition of a Poincaré transformation and two rotations. Since $S_v \circ\ E_v$ and $E_u^{-1} \circ S_u^{-1}$ are Lorentz transformations (which are special cases of Poincaré transformations),

$$T_2 = E_u^{-1} \circ\ S_u^{-1} \circ T_1 \circ S_v \circ E_v$$

is a Poincaré transformation.

$$T_3 = G_u^{-1} \circ T_2 \circ G_v$$

is a Poincaré transformation. T_3 is also a trivial transformation because it is a transformation between two primitive ether observers, which are at rest relative to each other by AxPrimitiveEther. Trivial Poincaré transformations are also (trivial) Galilean transformations. Consequently,

$$R_{\bar{u}}^{-1} \circ T_3 \circ R_{\bar{v}}$$

is a Galilean transformation because T_3 is a Galilean transformations and because rotations $R_{\bar{u}}^{-1}$ and $R_{\bar{v}}$ are trivial Galilean transformations. □

Lemma 7. Assume $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$. Let e be a primitive ether observer, and let k and h be inertial observers. Assume that w_{hk}^{SR} is a trivial transformation which is translation by vector $\bar{z}$ after linear trivial transformation T. Then $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{SR} \circ Rad_{\bar{v}_h(e)}$ is the trivial transformation which is translation by vector $Rad_{\bar{v}_k(e)}^{-1}(\bar{z})$ after T.

Proof. By Lemma 6, $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{SR} \circ Rad_{\bar{v}_h(e)}$ is a Galilean transformation. Since w_{hk}^{SR} is a trivial transformation, it maps vertical lines to vertical ones. By Lemma 1, $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{SR} \circ Rad_{\bar{v}_h(e)}$ also maps vertical lines to vertical ones. Consequently, $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{CK} \circ Rad_{\bar{v}_h(e)}$ is a Galilean transformation that maps vertical lines to vertical ones. Hence it is a trivial transformation.

Let $M_{\bar{z}}$ denote the translation by vector $\bar{z}$.

By the assumptions, $w_{hk}^{SR} = M_{\bar{z}} \circ T$. The linear part T of w_{hk}^{SR} transforms the velocity of the primitive ether frame as $(0, \bar{v}_k(e)) = T(0, \bar{v}_h(e))$ and the translation part $M_{\bar{z}}$ does not change the velocity of the primitive ether frame. Hence $\bar{v}_k(e)$ is $\bar{v}_h(e)$ transformed by the spatial isometry part of T.

We also have that $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{SR} \circ Rad_{\bar{v}_h(e)}$ is $Rad_{\bar{v}_k(e)}^{-1} \circ M_{\bar{z}} \circ T \circ Rad_{\bar{v}_h(e)}$. Since $Rad_{\bar{v}_k(e)}^{-1}$ is linear, we have $Rad_{\bar{v}_k(e)}^{-1} \circ M_{\bar{z}} = M_{Rad_{\bar{v}_k(e)}^{-1}(\bar{z})} \circ Rad_{\bar{v}_k(e)}^{-1}$. Therefore, it is enough to prove that $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{SR} \circ Rad_{\bar{v}_h(e)} = w_{hk}^{SR}$ if w_{hk}^{SR} is linear.

From now on, assume that w_{hk}^{CK} is linear.

Since it is a linear trivial transformation, w_{hk}^{SR} maps $(1, 0, 0, 0)$ to it self. By Item 3 of Lemma 1, $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{SR} \circ Rad_{\bar{v}_h(e)} = w_{hk}^{CK}$ also maps $(1, 0, 0, 0)$ to it self because $Rad_{\bar{v}_h(e)}$ scales up the time axis the same factor as $Rad_{\bar{v}_k(e)}^{-1}$ scales down because $|\bar{v}_h(e)| = |\bar{v}_k(e)|$. So $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{SR} \circ Rad_{\bar{v}_h(e)}$ and w_{hk}^{SR} agree restricted to time.

Now we have to prove that $Rad_{\bar{v}_k(e)}^{-1} \circ w_{hk}^{SR} \circ Rad_{\bar{v}_h(e)}$ and w_{hk}^{SR} also agree restricted to space. By Item 5 of Lemma 1, $Rad_{\bar{v}_h(e)}$ is identical on the vectors orthogonal to the plane containing the time-axis and the direction of motion of the ether frame, determined by vector $\bar{v}_h(e)$. Worldview transformation w_{hk}^{SR} leaves the time-axis fixed and maps the velocity of ether frame $\bar{v}_h(e)$ to $\bar{v}_k(e)$. After this $Rad_{\bar{v}_k(e)}^{-1}$ does not change the vectors orthogonal to the plane containing the time-axis and the direction of motion of the ether frame. Therefore, $Rad^{-1}_{\bar{v}_k(e)} \circ w^{SR}_{hk} \circ Rad_{\bar{v}_h(e)}$ and w^{SR}_{hk} do the same thing with the vectors orthogonal to the plane containing the time-axis and the direction of motion of the ether frame, determined by vector $\bar{v}_h(e)$. So the space part of $Rad^{-1}_{\bar{v}_k(e)} \circ w^{SR}_{hk} \circ Rad_{\bar{v}_h(e)}$ and w^{SR}_{hk} are isometries of Q^3 that agree on two independent vectors. This means that they are either equal or differ in a mirroring. However, they cannot differ in a mirroring as $Rad^{-1}_{\bar{v}_k(e)}$ and $Rad_{\bar{v}_h(e)}$ are orientation preserving maps because all of their components are such. Consequently, $Rad^{-1}_{\bar{v}_k(e)} \circ w^{SR}_{hk} \circ Rad_{\bar{v}_h(e)} = w^{SR}_{hk}$. □

Theorem 7. Tr'_+ is an interpretation of $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ to $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$, i.e.,

$$\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_+(\varphi) \quad \text{if} \quad \mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash \varphi.$$

Proof.

- $\mathsf{EField}^{\mathsf{SR}} \vdash Tr'_+(\mathsf{AxEField}^{\mathsf{CK}})$ since mathematical formulas are translated into themselves.

- Let us now prove that $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_+(\mathsf{AxEv}^{\mathsf{CK}})$. The translation of $\mathsf{AxEv}^{\mathsf{CK}}$ is equivalent to:

$$(\forall k, h \in IOb)\big(\forall \bar{x} \in Q^4\big)(\forall e \in E)\big(\exists \bar{y} \in Q^4\big)$$
$$\Big[ev_k\left(Rad_{\bar{v}_k(e)}(\bar{x})\right) = ev_h\left(Rad_{\bar{v}_h(e)}(\bar{y})\right)\Big].$$

  To prove the formula above, let k and h be inertial observers, let e be a primitive ether observer, and let $\bar{x} \in Q^4$. We have to prove that there is a $\bar{y} \in Q^4$ such that $ev_k[Rad_{\bar{v}_k(e)}(\bar{x})] = ev_h[Rad_{\bar{v}_h(e)}(\bar{y})]$. Let us denote $Rad_{\bar{v}_k(e)}(\bar{x})$ by $\bar{x}'$. $\bar{x}'$ exists since $Rad_{\bar{v}_k(e)}$ is a well-defined bijection. There is a $\bar{y}'$ such that $ev_k(\bar{x}') = ev_h(\bar{y}')$ because of $\mathsf{AxEv}^{\mathsf{SR}}$. Then $\bar{y} = Rad^{-1}_{\bar{v}_h(e)}(\bar{y}')$ has the requited properties.

- Let us now prove that $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_+(\mathsf{axAxLine}^{\mathsf{CK}})$. The translation of $\mathsf{AxLine}^{\mathsf{CK}}$ is equivalent to:

$$(\forall k, h \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{z} \in Q^4\big)\big(\forall e \in Ether\big)$$
$$\Big[Rad_{\bar{v}_k(e)}(\bar{x}), Rad_{\bar{v}_k(e)}(\bar{y}), Rad_{\bar{v}_k(e)}(\bar{z}) \in wl_k(h)$$
$$\to (\exists a \in Q)\big[\bar{z} - \bar{x} = a(\bar{y} - \bar{x}) \lor \bar{y} - \bar{z} = a(\bar{z} - \bar{x})\big]\Big].$$

  Because of $\mathsf{AxLine}^{\mathsf{SR}}$, $Rad_{\bar{v}_k(e)}(\bar{x})$, $Rad_{\bar{v}_k(e)}(\bar{y})$ and $Rad_{\bar{v}_k(e)}(\bar{z})$ are on a straight line. Since $Rad_{\bar{v}}$ is a linear map, $\bar{x}$, $\bar{y}$ and $\bar{z}$ are on a straight line, hence the translation of $\mathsf{AxLine}^{\mathsf{CK}}$ follows.

- Let us now prove that $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_+(\mathsf{AxSelf}^{\mathsf{CK}})$. The translation of $\mathsf{AxSelf}^{\mathsf{CK}}$ is equivalent to

$$(\forall k \in IOb)(\forall e \in E)\big(\forall \bar{y} \in Q^4\big)\\ \big[W\big((k,k,Rad_{\bar{v}_k(e)}(\bar{y})\big) \leftrightarrow y_1 = y_2 = y_3 = 0\big].$$

To prove the formula above, let k be an inertial observer, let e be a primitive ether observer, and let $\bar{y} \in Q^4$. We have to prove that $W\big((k,k,Rad_{\bar{v}_k(e)}(\bar{y})\big)$ if and only if $y_1 = y_2 = y_3 = 0$. Let $\bar{x} \in Q^4$ be such that $Rad^{-1}_{\bar{v}_k(e)}(\bar{x}) = \bar{y}$. By $\mathsf{AxSelf}^{\mathsf{SR}}$, $W\big((k,k,\bar{x})\big)$ if and only if $x_1 = x_2 = x_3 = 0$. This holds if and only if $y_1 = y_2 = y_3 = 0$ since by item 2 of Lemma 1 $Rad_{\bar{v}}$ transformation maps the time-axis to the time-axis.

- Let us now prove that $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_+(\mathsf{AxSymD}^{\mathsf{CK}})$. The translation of $\mathsf{AxSymD}^{\mathsf{CK}}$ is equivalent to:

$$(\forall k,k' \in IOb)\big(\forall \bar{x},\bar{y},\bar{x}',\bar{y}' \in Q^4\big)(\forall e \in E)\\ \left(\begin{bmatrix} time(\bar{x},\bar{y}) = time(\bar{x}',\bar{y}') = 0 \\ ev_k(Rad_{\bar{v}_k(e)}(\bar{x})) = ev_{k'}(Rad_{\bar{v}_{k'}(e)}(\bar{x}')) \\ ev_k(Rad_{\bar{v}_k(e)}(\bar{y})) = ev_{k'}(Rad_{\bar{v}_{k'}(e)}(\bar{y}')) \end{bmatrix} \\ \rightarrow space(\bar{x},\bar{y}) = space(\bar{x}',\bar{y}')\right).$$

Let k and k' be inertial observers, let $\bar{x}$, $\bar{y}$, $\bar{x}'$, and $\bar{y}'$ be coordinate points, and let e be a primitive ether observer such that $time(\bar{x},\bar{y}) = time(\bar{x}',\bar{y}') = 0$, $ev_k(Rad_{\bar{v}_k(e)}(\bar{x})) = ev_{k'}(Rad_{\bar{v}_{k'}(e)}(\bar{x}'))$, and $ev_k(Rad_{\bar{v}_k(e)}(\bar{y})) = ev_{k'}(Rad_{\bar{v}_{k'}(e)}(\bar{y}'))$. Let $G = Rad^{-1}_{v_{k'}(e)} \circ w_{kk'} \circ Rad_{v_k(e)}$. By Lemma 6, G is a Galilean transformation. By the assumptions, $G(\bar{x}) = \bar{x}'$ and $G(\bar{y}) = \bar{y}'$. Since G is a Galilean transformation and $time(\bar{x},\bar{y}) = 0$ we have $space(\bar{x},\bar{y}) = space(\bar{x}',\bar{y}')$.

- Let us now prove that $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_+(\mathsf{AxTriv}^{\mathsf{CK}})$. The translation of $\mathsf{AxTriv}^{\mathsf{CK}}$ is equivalent to:

$$(\forall T \in Triv)(\forall h \in IOb)(\forall e \in E)\\ \big((\exists k \in IOb) Rad^{-1}_{\bar{v}_k(e)} \circ w_{hk} \circ Rad_{\bar{v}_h(e)} = T\big).$$

To prove $Tr'_+(\mathsf{AxTriv}^{\mathsf{CK}})$, we have to find an inertial observer k for every trivial transformation T and an inertial observer h such that $Rad^{-1}_{\bar{v}_k(e)} \circ w_{hk} \circ$

$Rad_{\bar{v}_h(e)} = T$. By $\mathsf{AxTriv^{SR}}$ and Lemma 7, there is an inertial observer k such that $Rad^{-1}_{\bar{v}_k(e)} \circ w_{hk} \circ Rad_{\bar{v}_h(e)} = T$.

- Let us now prove that $\mathsf{SpecRel^e_{Full}} \vdash Tr'_+(\mathsf{AxAbsTime^{CK}})$. The translation of $\mathsf{AxAbsTime^{CK}}$ is equivalent to:

$$(\forall k, k' \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4\big)(\forall e \in E)$$
$$\left(\begin{bmatrix} ev_k(Rad_{\bar{v}_k(e)}(\bar{x})) = ev_{k'}(Rad_{\bar{v}_{k'}(e)}(\bar{x}')) \\ ev_k(Rad_{\bar{v}_k(e)}(\bar{y})) = ev_{k'}(Rad_{\bar{v}_{k'}(e)}(\bar{y}')) \end{bmatrix} \to time(\bar{x},\bar{y}) = time(\bar{x}',\bar{y}')\right).$$

  Let k and k' be inertial observers, let $\bar{x}$, $\bar{y}$, $\bar{x}'$, and $\bar{y}'$ be coordinate points, and let e be a primitive ether observer such that $ev_k(Rad_{\bar{v}_k(e)}(\bar{x})) = ev_{k'}(Rad_{\bar{v}_{k'}(e)}(\bar{x}'))$, and $ev_k(Rad_{\bar{v}_k(e)}(\bar{y})) = ev_{k'}(Rad_{\bar{v}_{k'}(e)}(\bar{y}'))$. Let $G = Rad^{-1}_{v_{k'}(e)} \circ w_{kk'} \circ Rad_{v_k(e)}$. By Lemma 6, G is a Galilean transformation. By the assumptions, $G(\bar{x}) = \bar{x}'$ and $G(\bar{y}) = \bar{y}'$. Since G is a Galilean transformation, which keeps simultaneous events simultaneous, and $\mathsf{AxSymD^{SR}}$ we have $time(\bar{x},\bar{y}) = time(\bar{x}',\bar{y}')$.

- Let us now prove that $\mathsf{SpecRel^e_{Full}} \vdash Tr'_+(\mathsf{AxNoFTL})$. The translation of $\mathsf{AxNoFTL}$ is equivalent to:

$$(\forall k \in IOb)(\forall e \in E)\big[speed_e(k) < \mathfrak{c}\big].$$

  This follows from Corollary 4.

- Let us now prove that $\mathsf{SpecRel^e_{Full}} \vdash Tr'_+(\mathsf{AxExpTh^{STL}})$. The translation of $\mathsf{AxExpTh^{STL}}$ is equivalent to:

$$(\exists h \in B)[IOb(h)] \land$$
$$(\forall e \in E)\big(\forall \bar{x}, \bar{y} \in Q^4\big)\big(space(\bar{x},\bar{y}) < \mathfrak{c} \cdot time(\bar{x},\bar{y})$$
$$\to (\exists k \in IOb)\big[Rad_{\bar{v}_k(e)}(\bar{x}), Rad_{\bar{v}_k(e)}(\bar{y}) \in wl_e(k)\big]\big).$$

  From $\mathsf{AxThExp}$ we get inertial observers inside of the light cones. Inertial observers stay inside the light cone by the translation by Items 2 and 4 of Lemma 1.

- Let us now prove that $\mathsf{SpecRel^e_{Full}} \vdash Tr'_+(\mathsf{AxEther})$. The translation of $\mathsf{AxEther}$ is equivalent to:

$$(\exists e \in B)\big[E(e)\big].$$

  This follows from $\mathsf{AxPrimitiveEther}$.

- $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_{+}(\mathsf{AxNoAcc})$

  The translation of AxNoAcc is equivalent to:

$$(\forall k \in B)(\exists \bar{x} \in Q^4)(\exists b \in B)(\forall e \in Ether)\\ \big[W^{SR}\big(k, b, Rad_{\bar{v}_k(e)}(\bar{x})\big) \to IOb^{SR}(k)\big],$$

  which follows directly from AxNoAcc since $Rad_{\bar{v}_k(e)}$ is the same bijection for all ether observers e. □

Lemma 8. Both translations Tr_+ and Tr'_+ preserve the concept "ether velocity":

$$\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr_+\big(E(e) \wedge \bar{v}_k(e) = \bar{v}\big) \leftrightarrow Ether(e) \wedge \bar{v}_k(e) = \bar{v}$$
$$\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}} \vdash Tr'_+\big(Ether(e) \wedge \bar{v}_k(e) = \bar{v}\big) \leftrightarrow E(e) \wedge \bar{v}_k(e) = \bar{v}$$

Proof. The translation of $\bar{v}_k(b) = \bar{v}$ by Tr_+ is:

$$Tr_+\left(\left[\begin{array}{c} \big(\exists \bar{x}, \bar{y} \in wl_k(b)\big)(\bar{x} \neq \bar{y}) \\ \big(\forall \bar{x}, \bar{y} \in wl_k(b)\big)\left[(y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0)\right] \end{array}\right]\right),$$

which is equivalent to

$$(\forall e \in Ether)\left[\begin{array}{c} \big(\exists \bar{x}', \bar{y}' \in wl_k(b)\big)\big(Rad_{\bar{v}_k(e)}(\bar{x}') \neq Rad_{\bar{v}_k(e)}(\bar{y}')\big) \\ \big(\forall \bar{x}', \bar{y}' \in wl_k(b)\big)\left[\begin{array}{c} (y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0) \\ \bar{x} = Rad_{\bar{v}_k(e')}(\bar{x}') \\ \bar{y} = Rad_{\bar{v}_k(e')}(\bar{y}') \end{array}\right] \end{array}\right].$$

The translation of velocity relative to the primitive ether by Tr_+ is:

$$Tr_+[E(e) \wedge \bar{v}_k(e) = \bar{v}] \equiv Ether(e) \wedge$$
$$(\forall e' \in Ether)\left[\begin{array}{c} \big(\exists \bar{x}', \bar{y}' \in wl_k(e)\big)\big(Rad_{\bar{v}_k(e')}(\bar{x}') \neq Rad_{\bar{v}_k(e')}(\bar{y}')\big) \\ \big(\forall \bar{x}', \bar{y}' \in wl_k(e)\big)\left[\begin{array}{c} (y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0) \\ \bar{x} = Rad_{\bar{v}_k(e')}(\bar{x}') \\ \bar{y} = Rad_{\bar{v}_k(e')}(\bar{y}') \end{array}\right] \end{array}\right]$$

Since e' only occurs in $v_k(e')$ and since all ether observers are at rest relative to each other by Corollary 1, they all have the same speed relative to k. Since, by Lemma 1, $Rad_{\bar{v}}$ is a bijection, $Rad_{\bar{v}}(\bar{x}') \neq Rad_{\bar{v}}(\bar{y}')$ is equivalent to $\bar{x} \neq \bar{y}$. Hence we can simplify the above to:

$$Tr_+[E(e) \wedge \bar{v}_k(e) = \bar{v}] \equiv Ether(e) \wedge \big(\exists \bar{x}', \bar{y}' \in wl_k(e)\big)[\bar{x}' \neq \bar{y}'] \wedge$$
$$\big(\forall \bar{x}', \bar{y}' \in wl_k(e)\big)\left[\begin{array}{c} (y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0) \\ \bar{x} = Rad_{\bar{v}_k(e)}(\bar{x}') \wedge \bar{y} = Rad_{\bar{v}_k(e)}(\bar{y}') \end{array}\right],$$

which says that the $Rad_{\bar{v}_k(e)}$-image of worldline $wl_k(e)$ moves with speed $\bar{v}$, which is equivalent to that $wl_k(e)$ moves with speed $\bar{v}$ by Item 6 of Lemma 1, hence

$$Tr_+\big(E(e) \wedge \bar{v}_k(e) = \bar{v}\big) \equiv Ether(e) \wedge \bar{v}_k(e) = \bar{v}.$$

We will now prove this in the other direction. The translation of velocity by Tr'_+ is:

$$Tr'_+[\bar{v}_k(b) = \bar{v}] \equiv$$
$$(\forall e \in E)\left[\begin{array}{c} \big(\exists \bar{x}', \bar{y}' \in wl_k(b)\big)\big(Rad^{-1}_{\bar{v}_k(e)}(\bar{x}') \neq Rad^{-1}_{\bar{v}_k(e)}(\bar{y}')\big) \\ \big(\forall \bar{x}', \bar{y}' \in wl_k(b)\big)\left[\begin{array}{c} (y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0) \\ \bar{x} = Rad^{-1}_{\bar{v}_k(e')}(\bar{x}') \\ \bar{y} = Rad^{-1}_{\bar{v}_k(e')}(\bar{y}') \end{array}\right] \end{array}\right].$$

The translation of velocity relative to the ether by Tr'_+ is equivalent to:

$$Tr'_+[Ether(e) \wedge \bar{v}_k(e) = \bar{v}] \equiv$$
$$E(e) \wedge \big(\exists \bar{x}', \bar{y}' \in wl_k(e)\big)[\bar{x}' \neq \bar{y}'] \wedge$$
$$\big(\forall \bar{x}', \bar{y}' \in wl_k(e)\big)\left[\begin{array}{c} (y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0) \\ \bar{x} = Rad^{-1}_{\bar{v}_k(e)}(\bar{x}') \wedge \bar{y} = Rad^{-1}_{\bar{v}_k(e)}(\bar{y}') \end{array}\right],$$

which by $Rad_{\bar{v}_k(e)}$ being a bijection and Item 6 of Lemma 1 leads us to

$$Tr'_+\big(Ether(e) \wedge \bar{v}_k(e) = \bar{v}\big) \equiv E(e) \wedge \bar{v}_k(e) = \bar{v}.$$ □

Theorem 8. Tr_+ is a definitional equivalence between theories $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ and $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$.

Proof. We only need to prove that the inverse translations of the translated statements are logical equivalent to the original statements since Tr_+ and Tr'_+ are interpretations by Theorem 6 and Theorem 7. AxNoAcc guarantees that $\bar{v}_k(e)$ is defined for every ether observer e and observer k.

- Mathematical expressions, quantities and light signals are translated into themselves by both Tr_+ and Tr'_+.
- $Tr'_+\big(Tr_+[E(e)]\big) \equiv Tr'_+[Ether(e)] \equiv E(e)$ follows from the definition of Tr_+ and Lemma 5.
- The back and forth translation of IOb^{SR} is the following:

$$Tr'_+\big(Tr_+[IOb^{SR}(k)]\big)$$
$$\equiv Tr'_+\big(IOb^{CK}(k) \wedge (\forall e \in Ether)\big[speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big]\big)$$
$$\equiv IOb^{SR}(k) \wedge (\forall e \in E)\big[speed_e(k) < \mathfrak{c}\big] \equiv IOb^{SR}(k).$$

The second equivalence is true because of Lemma 5 and Corollary 6. The last equivalence is true because observers are always slower-than-light in $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$, as per Corollary 4.

- The back and forth translation of IOb^{CK} is the following:

$$Tr_+\big(Tr'_+[IOb^{CK}(b)]\big) \equiv Tr_+[IOb^{SR}(b)] \\ \equiv IOb^{CK}(b) \land (\forall e \in Ether)\big[speed_e(b) < \mathfrak{c}_\mathfrak{e}\big] \equiv IOb^{CK}(b).$$

The last equivalence holds because $(\forall e \in Ether)\big[speed_e(b) < \mathfrak{c}_\mathfrak{e}\big]$ is true by AxNoFTL.

- The back and forth translation of W^{SR} is the following:

$$Tr'_+\big(Tr_+[W^{SR}(k,b,\bar{x})]\big) \equiv Tr\Big[(\forall e \in Ether)\big[W^{CK}\big(k,b,Rad^{-1}_{\bar{v}_k(e)}(\bar{x})\big)\big]\Big] \\ \equiv (\forall e \in E)\Big(W^{SR}\big[k,b,Rad_{\bar{v}_k(e)}\big(Rad^{-1}_{\bar{v}_k(e)}(\bar{x})\big)\big]\Big) \\ \equiv (\forall e \in E)[W^{SR}(k,b,\bar{x})] \equiv W^{SR}(k,b,\bar{x}).$$

We use Lemma 8 to translate the indexes $\bar{v}_k(e)$ into themselves.

- The back and forth translation of W^{CK} is the following:

$$Tr_+\big(Tr'_+[W^{CK}(k,b,\bar{x})]\big) \equiv Tr_+\Big[(\forall e \in E)\big[W^{SR}\big(k,b,Rad_{\bar{v}_k(e)}(\bar{x})\big)\big]\Big] \\ \equiv (\forall e \in Ether)\Big(W^{SR}\big[k,b,Rad^{-1}_{\bar{v}_k(e)}\big(Rad_{\bar{v}_k(e)}(\bar{x})\big)\big]\Big) \\ \equiv (\forall e \in Ether)[W^{CK}(k,b,\bar{x})] \equiv W^{CK}(k,b,\bar{x}).$$

We use Lemma 8 to translate the indexes $\bar{v}_k(e)$ into themselves. □

9. FTL OBSERVERS ARE DEFINABLE FROM STL ONES IN CLASSICAL KINEMATICS

In this chapter, where we only work with classical theories, we show that $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ and $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$, which is $\mathsf{ClassicalKin}_{\mathsf{Full}} \cup \{\mathsf{AxNoAcc}\}$, are definitionally equivalent theories.

We can map the interval of speeds $[0, \mathfrak{c}_\mathfrak{e}]$ to $[0, \infty]$ by replacing slower-than-light speed v by classical speed $V = \frac{v}{\mathfrak{c}_\mathfrak{e} - v}$, and conversely map the interval of speeds $[0, \infty]$ to $[0, \mathfrak{c}_\mathfrak{e}]$ by replacing speed V by speed $v = \frac{\mathfrak{c}_\mathfrak{e} V}{1+V}$.

Similarly, for arbitrary (finite) velocity $\bar{V}$, we have that $\bar{v} = \frac{\mathfrak{c}_\mathfrak{e} \bar{V}}{1+|\bar{V}|}$ is slower than $\mathfrak{c}_\mathfrak{e}$, and from $\bar{v}$, we can get $\bar{V}$ back by equation $\bar{V} = \frac{\bar{v}}{\mathfrak{c}_\mathfrak{e} - |\bar{v}|}$.

Definition 27. Let $G_{\bar{V}}$ and $G_{\bar{v}}$, respectively, be the Galilean boosts that map bodies moving with velocity $\bar{V}$ and $\bar{v}$ to stationary ones. Let $X_{\bar{V}} = G_{\bar{v}}^{-1} \circ G_{\bar{V}}$ and $Y_{\bar{v}} = G_{\bar{V}}^{-1} \circ G_{\bar{v}}$, see Figure 9.

Lemma 9. Let $\bar{v} \in Q^3$ for which $|\bar{v}| \in [0, \mathfrak{c}_\mathfrak{e}]$ and let $\bar{V} = \frac{\bar{v}}{\mathfrak{c}_\mathfrak{e} - |\bar{v}|}$. Then $X_{\bar{V}}^{-1} = Y_{\bar{v}}$.

Proof. By definition of $X_{\bar{V}}$ and $Y_{\bar{v}}$ and by the inverse of composed transformations:

$$X_{\bar{V}}^{-1} = \left(G_{\bar{v}}^{-1} \circ G_{\bar{V}}\right)^{-1} = G_{\bar{V}}^{-1} \circ \left(G_{\bar{v}}^{-1}\right)^{-1} = G_{\bar{V}}^{-1} \circ G_{\bar{v}} = Y_{\bar{v}}. \qquad \square$$

Assume that the ether frame is moving with a faster than light velocity $\bar{V}$ with respect to an inertial observer k. Then by $X_{\bar{V}}$ we can transform the worldview of k such a way that after the transformation the ether frame is moving slower than light with respect to k, see Figure 9.3. Systematically modifying every observer's worldview using the corresponding transformation $X_{\bar{V}}$, we can achieve the result that every observer see that the ether frame is moving slower than light. These transformations tell us where the observers should see the non-observer bodies. However, this method is usually not working for bodies representing inertial observers because transformation $X_{\bar{V}}$ leave the time-axis fixed only if $\bar{V} = (0, 0, 0)$. Therefore, we will translate the worldlines of bodies representing observers in harmony to AxSelf to represent the motion of the corresponding observer's coordinate system.

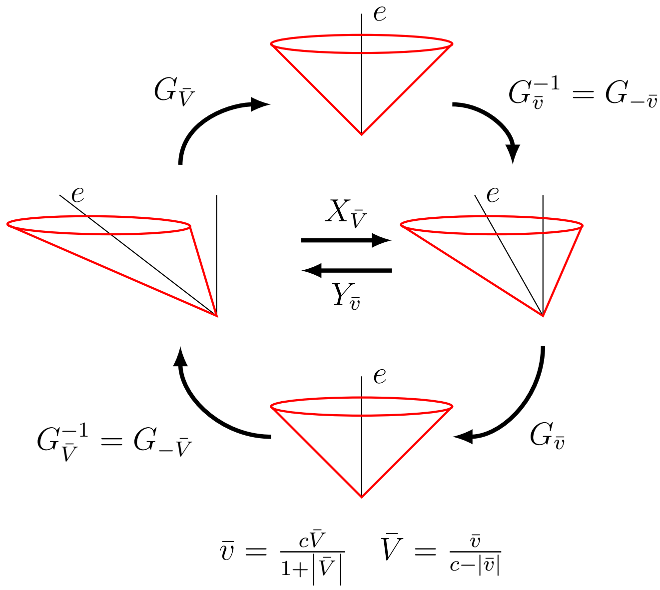

Fig. 9.1: $\bar V$ is an arbitrary velocity and $\bar v = \frac{c\bar V}{1+|\bar V|}$ is the corresponding STL velocity. $G_{\bar V}$ and $G_{\bar v}$ are Galilean boosts that, respectively, map bodies moving with velocity $\bar V$ and $\bar v$ to stationary ones. Transformations $X_{\bar V} = G_{\bar v}^{-1} \circ G_{\bar V}$ and $Y_{\bar v} = G_{\bar V}^{-1} \circ G_{\bar v}$ allow to map between observers seeing the ether frame moving with up to infinite speeds on the left and STL speeds on the right. The light cones on the top and on the bottom are the same one.

This means that we have to split up the translation of W between the observer and non-observer cases.[15]

Definition 28. $X_{\bar v_k(e)}(\bar x)$ and $Y_{\bar v_k(e)}(\bar x)$ and their inverses are:

$$X_{\bar v_k(e)}(\bar x) = \bar y \overset{\text{def}}{\iff} X^{-1}_{\bar v_k(e)}(\bar y) = \bar x \overset{\text{def}}{\iff} \big(\exists \bar v \in Q^3\big)[\bar V = \bar v_k(e) \land X_{\bar V}(\bar x) = \bar y],$$

and

$$Y_{\bar v_k(e)}(\bar x) = \bar y \overset{\text{def}}{\iff} Y^{-1}_{\bar v_k(e)}(\bar y) = \bar x \overset{\text{def}}{\iff} \big(\exists \bar v \in Q^3\big)[\bar v = \bar v_k(e) \land Y_{\bar v}(\bar x) = \bar y].$$

[15] There are other ways to handle this issue, such as introducing a new sort for inertial observers. This however would complicate the previous chapters of this dissertation, and also take our axiom system further away from stock SpecRel.

Definition 29. Tr_* is the following translation:

$$Tr_*\big(W^{STL}(k,b,\bar{x})\big) \stackrel{\text{def}}{\equiv} (\forall e \in Ether)$$
$$\left[\begin{array}{c} b \notin IOb \rightarrow W^{CK}\big(k,b,X^{-1}_{\bar{v}_k(e)}(\bar{x})\big) \\ b \in IOb \rightarrow (\exists t \in Q)\big[w^{CK}_{kb}\big(X^{-1}_{\bar{v}_k(e)}(\bar{x})\big) = X^{-1}_{\bar{v}_b(e)}(t,0,0,0)\big] \end{array}\right].$$

and Tr_* is identical on the other concepts.

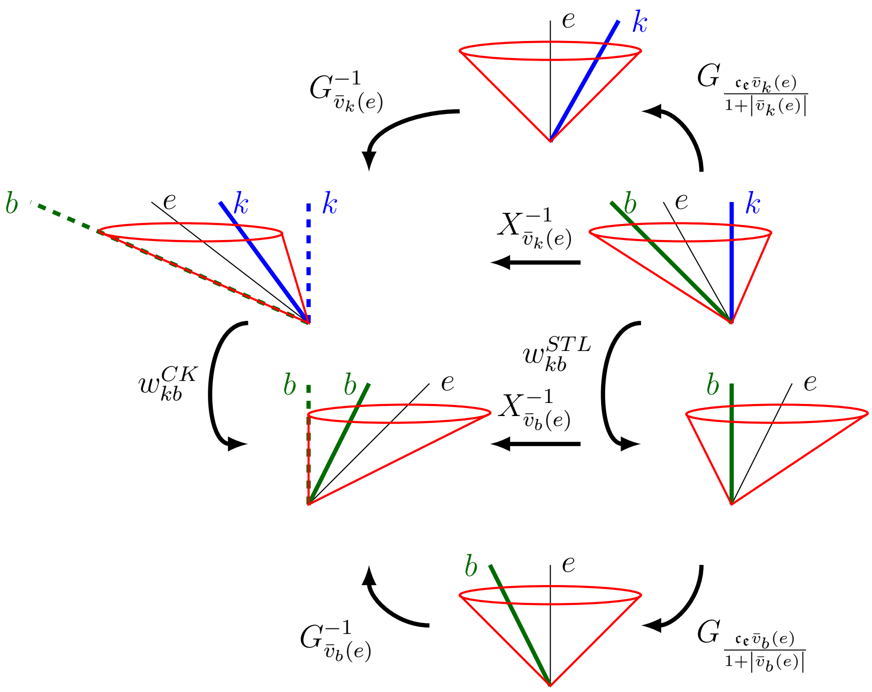

Fig. 9.2: Transformations from on the right, $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ to, on the left, $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$ for on top, observer k and, at the bottom, observer b. The dotted lines are where we put the observers after the transformation in order to respect AxSelf. All transformations, including the worldviev transfrormations from the top to the bottom, are Galilean.

Definition 30. Tr'_* is the following translation:

$$Tr'_*\big(W^{CK}(k,b,\bar{x})\big) \stackrel{\text{def}}{\equiv} (\forall e \in Ether)$$
$$\left[\begin{array}{c} b \notin IOb \rightarrow W^{STL}\big(k,b,Y^{-1}_{\bar{v}_k(e)}(\bar{x})\big) \\ b \in IOb \rightarrow (\exists t \in Q)\big[w^{STL}_{kb}\big(Y^{-1}_{\bar{v}_k(e)}(\bar{x})\big) = Y^{-1}_{\bar{v}_b(e)}(t,0,0,0)\big] \end{array}\right].$$

and Tr'_* is the identity on the other concepts.

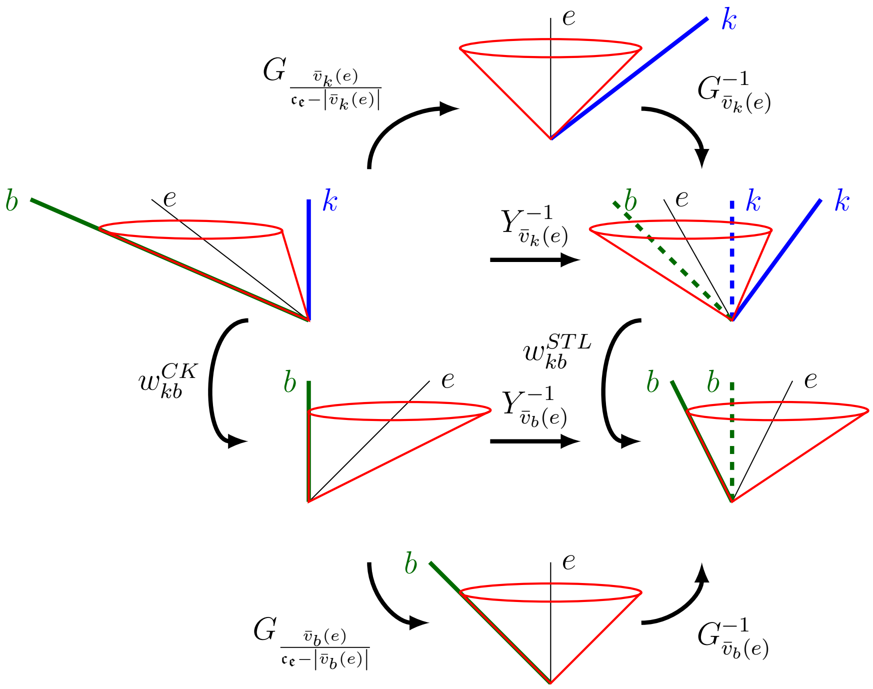

Fig. 9.3: Transformations from, on the left, $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$ to, on the right, $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ for on top, observer k and, at the bottom, observer b. The dotted lines are where we put the observers after the transformation in order to respect AxSelf. All transformations, including the worldviev transfromations from the top to the bottom, are Galilean. The cone at the bottom is only displayed for completion, it is not being used in our reasoning.

Lemma 10. Assume $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$. Then

$$Tr'_*\left(w^{CK}_{kh}\right) = Y_{\bar{v}_h(e)} \circ w^{STL}_{kh} \circ Y^{-1}_{\bar{v}_k(e)}.$$

Proof. We should show that

$$(\forall b \in B)\big[Tr_*\big(W^{CK}(k,b,\bar{x})\big) \leftrightarrow Tr_*\big(W^{CK}(h,b,\bar{y})\big)\big] \tag{9.1}$$

holds iff

$$(\forall b \in B)(\forall e \in Ether)\big[w^{STL}_{kh}\big(Y^{-1}_{\bar{v}_k(e)}(\bar{x})\big) = Y^{-1}_{\bar{v}_h(e)}(\bar{y})\big] \tag{9.2}$$

holds.

Let us first assume that (9.1) holds. This is equivalent to conjunction of the following two things:

$$(\forall b \in B \setminus IOb)(\forall e \in Ether)$$
$$\big[\big(W^{STL}(k,b,Y^{-1}_{\bar{v}_k(e)}(\bar{x}))\big) \leftrightarrow \big(W^{STL}(h,b,Y^{-1}_{\bar{v}_h(e)}(\bar{y}))\big)\big] \tag{9.3}$$

and

$$(\forall b \in IOb)(\forall e \in Ether)\Big[(\exists t \in Q)\big[w^{STL}_{kb}\big(Y^{-1}_{\bar v_k(e)}(\bar x)\big) = Y^{-1}_{\bar v_b(e)}(t,0,0,0)\big] \leftrightarrow (\exists t \in Q)\big[w^{STL}_{hb}\big(Y^{-1}_{\bar v_h(e)}(\bar y)\big) = Y^{-1}_{\bar v_b(e)}(t,0,0,0)\big]\Big] \quad (9.4)$$

hold. By (9.3), we have that $W^{STL}\big(k,b,Y^{-1}_{\bar v_k(e)}(\bar x)\big)$ iff $W^{STL}\big(h,b,Y^{-1}_{\bar v_h(e)}(\bar y)\big)$ if b is not an inertial observer. To show (9.2), we have to show this also if b is an inertial observer. Let b be an inertial observer, then $W^{STL}\big(k,b,Y^{-1}_{\bar v_k(e)}(\bar x)\big)$ holds exactly if $w^{STL}_{kb}\big(Y^{-1}_{\bar v_k(e)}(\bar x)\big)$ is on the $Y^{-1}_{\bar v_b(e)}$-image of the time-axis by the definition of Tr_*. By (9.4), we have that $w^{STL}_{kb}\big(Y^{-1}_{\bar v_k(e)}(\bar x)\big)$ is on the $Y^{-1}_{\bar v_b(e)}$-image of the time-axis iff $w^{STL}_{hb}\big(Y^{-1}_{\bar v_h(e)}(\bar y)\big)$ is on the $Y^{-1}_{\bar v_b(e)}$-image of the time-axis, which is equivalent to $W^{STL}\big(h,b,Y^{-1}_{\bar v_h(e)}(\bar x)\big)$ by the definition of Tr'_*. Consequently, $W^{STL}\big(k,b,Y^{-1}_{\bar v_k(e)}(\bar x)\big)$ iff $W^{STL}\big(h,b,Y^{-1}_{\bar v_h(e)}(\bar y)\big)$. Therefore, (9.1) implies (9.2).

Let us now assume that (9.2) holds. Then (9.3) follows immediately from the definition of worldview transformation. By (9.2), we have $Y^{-1}_{\bar v_h(e)}(\bar y) = w^{STL}_{kh}\big(Y^{-1}_{\bar v_k(e)}(\bar x)\big)$. Hence, by the definition of worldview transformation,

$$w^{STL}_{hb}\big(Y^{-1}_{\bar v_h(e)}(\bar y)\big) = w^{STL}_{hb}\Big(w^{STL}_{kh}\big(Y^{-1}_{\bar v_k(e)}(\bar x)\big)\Big) = w^{STL}_{kb}\big(Y^{-1}_{\bar v_k(e)}(\bar x)\big),$$

which implies (9.4) immediately. □

Lemma 11. Assume $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$. Then

$$Tr_*\left(w^{STL}_{kb}\right) = X_{\bar v_b(e)} \circ w^{CK}_{kb} \circ X^{-1}_{\bar v_k(e)}.$$

Proof. The proof is completely analogous to that of Lemma 10. □

Lemma 12. Assume $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$. Let e be an ether observer, and let h and k be inertial observers. Assume that w^{CK}_{hk} is a trivial transformation which is translation by vector $\bar z$ after linear trivial transformation T_0. Then $X_{\bar v_k(e)} \circ w^{CK}_{hk} \circ X^{-1}_{\bar v_h(e)}$ is the trivial transformation which is translation by vector $X_{\bar v_k(e)}(\bar z)$ after T_0.

Proof. Let $M_{\bar z}$ denote the translation by vector $\bar z$.

By the assumptions, $w^{CK}_{hk} = M_{\bar z} \circ T_0$. The linear part T_0 of w^{CK}_{hk} transforms the velocity of the ether frame as $(0, \bar v_k(e)) = T_0(0, \bar v_h(e))$ and the translation part $M_{\bar z}$ does not change the velocity of the ether frame. Hence $\bar v_k(e)$ is $\bar v_h(e)$ transformed by the spatial isometry part of T_0.

By the definition of $X_{\bar v}$ and Theorem 2, $X_{\bar v_k(e)} \circ w^{SR}_{hk} \circ X^{-1}_{\bar v_h(e)}$ is a Galilean transformation. Since w^{CK}_{hk} is a trivial transformation, it maps vertical lines to vertical ones. $X_{\bar v_k(e)} \circ w^{CK}_{hk} \circ X^{-1}_{\bar v_h(e)}$ also maps vertical lines to vertical ones. Consequently,

$X_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ X_{\bar{v}_h(e)}^{-1}$ is a Galilean transformation that maps vertical lines to vertical ones. Hence it is a trivial transformation.

We also have that $X_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ X_{\bar{v}_h(e)}^{-1}$ is $X_{\bar{v}_k(e)} \circ M_{\bar{z}} \circ T_0 \circ X_{\bar{v}_h(e)}^{-1}$. Since $X_{\bar{v}_k(e)}$ is linear, we have $X_{\bar{v}_k(e)} \circ M_{\bar{z}} = M_{X_{\bar{v}_k(e)}(\bar{z})} \circ X_{\bar{v}_k(e)}$. Therefore, it is enough to prove that $X_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ X_{\bar{v}_h(e)}^{-1} = w_{hk}^{CK}$ if w_{hk}^{CK} is linear.

From now on assume that w_{hk}^{CK} is linear.

Since it is a linear trivial transformation, w_{hk}^{CK} maps $(1, 0, 0, 0)$ to it self. $X_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ X_{\bar{v}_h(e)}^{-1} = w_{hk}^{CK}$ also maps $(1, 0, 0, 0)$ to it self because $\bar{v}_k(e)$ is $\bar{v}_h(e)$ transformed by the spatial isometry part of w_{hk}^{CK}. So $X_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ X_{\bar{v}_h(e)}^{-1}$ and w_{hk}^{CK} also agree restricted to time.

Now we have to prove that $X_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ X_{\bar{v}_h(e)}^{-1}$ and w_{hk}^{CK} also agree restricted to space. Since they are Galilean boosts, $X_{\bar{v}_h(e)}$ and $X_{\bar{v}_k(e)}$ are identical on the vectors orthogonal to the time-axis. Consequently, $X_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ X_{\bar{v}_h(e)}^{-1}$ and w_{hk}^{CK} also agree restricted to space. Hence $X_{\bar{v}_k(e)} \circ w_{hk}^{CK} \circ X_{\bar{v}_h(e)}^{-1} = w_{hk}^{CK}$. □

Lemma 13.

$$\mathsf{ClassicalKin}_{\mathsf{Full}}^{\mathsf{NoAcc}} \vdash Tr_*(Ether(e)) \leftrightarrow Ether(e).$$

Proof. Assuming $\mathsf{ClassicalKin}_{\mathsf{Full}}^{\mathsf{NoAcc}}$, $Tr_*(Ether(e))$ is equivalent to

$$\Big(IOb(e) \wedge (\exists c \in Q)\Big[c > 0 \wedge \big(\forall \bar{x}, \bar{y} \in Q^4\big)(\forall e' \in Ether)$$
$$\Big((\exists p \in Ph)\begin{bmatrix} W^{CK}\big(e, p, X_{v_e(e')}^{-1}(\bar{x})\big) \\ W^{CK}\big(e, p, X_{v_e(e')}^{-1}(\bar{y})\big) \end{bmatrix} \leftrightarrow space(\bar{x}, \bar{y}) = c \cdot time(\bar{x}, \bar{y})\Big)\Big]\Big),$$

which, since by Corollary 1 all ether observers are stationary relative to each other and therefore $\big(\forall k \in IOb\big)\big(\forall e_1, e_2 \in Ether\big)[v_k(e_1) = v_k(e_2)]$, can be simplified to

$$\Big(IOb(e) \wedge (\exists c \in Q)\Big[c > 0 \wedge \big(\forall \bar{x}, \bar{y} \in Q^4\big)$$
$$\Big((\exists p \in Ph)\begin{bmatrix} W^{CK}\big(e, p, X_{\bar{v}_e(e)}^{-1}(\bar{x})\big) \\ W^{CK}\big(e, p, X_{\bar{v}_e(e)}^{-1}(\bar{y})\big) \end{bmatrix} \leftrightarrow space(\bar{x}, \bar{y}) = c \cdot time(\bar{x}, \bar{y})\Big)\Big]\Big).$$

Since $\bar{v}_e(e) = \bar{0}$, $X_{\bar{0}}^{-1}$ is by definition the identity, hence from $\mathsf{AxEther}^{\mathsf{CK}}$ follows that $Tr_*(Ether(e)) \leftrightarrow Ether(e)$. □

Corollary 7.

$$\mathsf{ClassicalKin}_{\mathsf{Full}}^{\mathsf{NoAcc}} \vdash Tr_*(\mathfrak{c}_{\mathfrak{e}}) = \mathfrak{c}_{\mathfrak{e}}.$$

Proof. This is a direct consequence of Lemma 13. □

Lemma 14.

$$\mathsf{ClassicalKin}_{\mathsf{Full}}^{\mathsf{STL}} \vdash Tr'_*(Ether(e)) \leftrightarrow Ether(e).$$

Proof. Assuming $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$, $Tr'_*(Ether(e))$ is, after simplification by Corollary 1 equivalent to

$$\Big(IOb(e) \wedge (\exists c \in Q)\Big[c > 0 \wedge \big(\forall \bar{x}, \bar{y} \in Q^4\big) \\ \big((\exists p \in Ph) \begin{bmatrix} W^{CK}\big(e, p, Y^{-1}_{\bar{v}_e(e)}(\bar{x})\big) \\ W^{CK}\big(e, p, Y^{-1}_{\bar{v}_e(e)}(\bar{y})\big) \end{bmatrix} \leftrightarrow space(\bar{x}, \bar{y}) = c \cdot time(\bar{x}, \bar{y})\big)\Big]\Big).$$

Since $\bar{v}_e(e) = \bar{0}$, $Y^{-1}_{\bar{0}}$ is by definition the identity, hence from $\mathsf{AxEther}^{\mathsf{STL}}$ follows that $Tr'_*(Ether(e)) \leftrightarrow Ether(e)$. □

Corollary 8.

$$\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr'_*(\mathfrak{c}_\mathfrak{e}) = \mathfrak{c}_\mathfrak{e}.$$

Proof. This is a direct consequence of Lemma 14. □

Lemma 15.

$$\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}} \vdash Tr_*\big(Ether(e) \wedge \bar{v}^{STL}_k(e) = \bar{v}\big) \\ \leftrightarrow \left(Ether(e) \wedge \bar{v}^{CK}_k(e) = \frac{\bar{v}}{\mathfrak{c}_\mathfrak{e} - |\bar{v}|}\right)$$

Proof. By Lemma 13, $Ether(e)$ translates into itself and hence

$$Tr_*\big(Ether(e) \wedge \bar{v}_k(e) = \bar{v}\big) \equiv Ether(e) \wedge \big(\forall e' \in Ether\big) \\ \left(\big(\exists \bar{x}, \bar{y} \in Q^4\big) \begin{bmatrix} (\exists \tau \in Q) \left(X_{\bar{v}_e(e')} \circ w^{CK}_{ke} \circ X^{-1}_{\bar{v}_k(e')}\right)(\bar{x}) = (\tau, 0, 0, 0) \\ (\exists \tau \in Q) \left(X_{\bar{v}_e(e')} \circ w^{CK}_{ke} \circ X^{-1}_{\bar{v}_k(e')}\right)(\bar{y}) = (\tau, 0, 0, 0) \\ \bar{x} \neq \bar{y} \end{bmatrix} \wedge \right. \\ \left. \big(\forall \bar{x}, \bar{y} \in Q^4\big) \begin{bmatrix} (\exists \tau \in Q) \left(X_{\bar{v}_e(e')} \circ w^{CK}_{ke} \circ X^{-1}_{\bar{v}_k(e')}\right)(\bar{x}) = (\tau, 0, 0, 0) \\ (\exists \tau \in Q) \left(X_{\bar{v}_e(e')} \circ w^{CK}_{ke} \circ X^{-1}_{\bar{v}_k(e')}\right)(\bar{y}) = (\tau, 0, 0, 0) \\ (y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0) \end{bmatrix} \right).$$

As $X_{\bar{v}_e(e)} = X_{\bar{0}} = Id$ and $\bar{v}_k(e') = \bar{v}_k(e)$ by Corollary 1, we can simplify this to

$$Tr_*\big(Ether(e) \wedge \bar{v}_k(e) = \bar{v}\big) \equiv Ether(e) \wedge \\ \big(\exists \bar{x}, \bar{y} \in Q^4\big) \begin{bmatrix} (\exists \tau \in Q) \left(w^{CK}_{ke} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = (\tau, 0, 0, 0) \\ (\exists \tau \in Q) \left(w^{CK}_{ke} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = (\tau, 0, 0, 0) \\ \bar{x} \neq \bar{y} \end{bmatrix} \wedge \\ \big(\forall \bar{x}, \bar{y} \in Q^4\big) \begin{bmatrix} (\exists \tau \in Q) \left(w^{CK}_{ke} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = (\tau, 0, 0, 0) \\ (\exists \tau \in Q) \left(w^{CK}_{ke} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = (\tau, 0, 0, 0) \\ (y_1 - x_1, y_2 - x_2, y_3 - x_3) = \bar{v} \cdot (y_0 - x_0) \end{bmatrix}.$$

Since $X^{-1}_{\bar{v}_k(e)} = G_{-\bar{v}_k(e)} \circ G_{\frac{\mathfrak{c}_\mathfrak{e}\bar{v}_k(e)}{1+|\bar{v}_k(e)|}}$, and from the images of $\bar{x}$ and $\bar{y}$ are on the time axis and hence by AxSelf are on the worldline of e according to e (meaning that in Figure 9.2 e coincides with b), $\bar{v} = \frac{\mathfrak{c}_\mathfrak{e}\bar{v}_k(e)}{1+|\bar{v}_k(e)|}$. From this it follows that the above is equivalent to

$$Ether(e) \wedge \bar{v}_k(e) = \frac{\bar{v}}{\mathfrak{c}_\mathfrak{e} - |\bar{v}|}. \qquad \square$$

Lemma 16.

$$\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr'_*\Big(Ether(e) \wedge \bar{v}^{CK}_k(e) = \bar{V}\Big) \leftrightarrow \left(Ether(e) \wedge \bar{v}^{STL}_k(e) = \frac{\mathfrak{c}_\mathfrak{e} \cdot \bar{V}}{1 + |\bar{V}|}\right)$$

Proof. The proof is analogous to the proof of Lemma 15. □

Theorem 9. Tr_* is an interpretation of $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ to $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$, i.e.,

$$\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}} \vdash Tr_*(\varphi) \quad \text{if} \quad \mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash \varphi.$$

Proof.

- $Tr_*(\mathsf{EField}^{\mathsf{STL}})$ follows from $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$ because $Tr_*(\mathsf{EField}^{\mathsf{STL}}) \equiv \mathsf{EField}^{\mathsf{CK}} \in \mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$.

- $Tr_*(\mathsf{AxSelf}^{\mathsf{STL}})$ is equivalent to the following formula:

$$(\forall k \in IOb)(\forall t, x, y, z \in Q)\Big((\forall e \in Ether)\; (\exists \tau \in Q)\big[X_{\bar{v}_k(e)}\big(w^{CK}_{kk}\big(X^{-1}_{\bar{v}_k(e)}(\bar{x})\big)\big) = (\tau, 0, 0, 0)\big] \leftrightarrow x = y = z = 0\Big).$$

  Since $w^{CK}_{kk} = Id$, $Tr_*(\mathsf{AxSelf}^{\mathsf{STL}})$ is equivalent to:

$$(\forall k \in IOb)(\forall t, x, y, z \in Q)\Big((\forall e \in Ether)\; (\exists \tau \in Q)\big[\bar{x} = (\tau, 0, 0, 0)\big] \leftrightarrow x = y = z = 0\Big)$$

  which is a tautology since $(t, x, y, z) = (\tau, 0, 0, 0)$ for some $\tau \in Q$ exactly if $x = y = z = 0$.

- By Lemma 11, $Tr_*(\mathsf{AxEv}^{\mathsf{STL}})$ is equivalent to the following formula:

$$(\forall k, h \in IOb)\big(\forall \bar{x} \in Q^4\big)\big(\exists \bar{y} \in Q^4\big)(\forall e \in Ether)\; \Big[\Big(X_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ X^{-1}_{\bar{v}_k(e)}\Big)(\bar{x}) = \bar{y}\Big].$$

Since $X_{\bar{v}_{k'}(e)} \circ w^{CK}_{kk'} \circ X^{-1}_{\bar{v}_k(e)}$ only consists of Galilean transformations, the composition is also a Galilean transformation. Since $X_{\bar{v}_{k'}(e)} \circ w^{CK}_{kk'} \circ X^{-1}_{\bar{v}_k(e)}$ is a Galilean transformation, a $\bar{y}$ for which $\left(X_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = \bar{y}$ must exist because Galilean transformations are surjective.

- By Lemma 11, $Tr_*(\mathsf{AxSymD}^{\mathsf{STL}})$, is equivalent to the following formula:

$$(\forall k, k' \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4\big)(\forall e \in Ether)$$
$$\left(\begin{bmatrix} time(\bar{x},\bar{y}) = time(\bar{x}',\bar{y}') = 0 \\ \left(X_{\bar{v}_{k'}(e)} \circ w^{CK}_{kk'} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = \bar{x}' \\ \left(X_{\bar{v}_{k'}(e)} \circ w^{CK}_{kk'} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = \bar{y}' \end{bmatrix} \to space(\bar{x},\bar{y}) = space(\bar{x}',\bar{y}')\right).$$

Since $\bar{x}$ and $\bar{y}$ are simultaneous before and after the Galilean transformation $X_{\bar{v}_{k'}(e)} \circ w^{CK}_{kk'} \circ X^{-1}_{\bar{v}_k(e)}$, their distance before and after the transformation must, by $\mathsf{AxSymD}^{\mathsf{CK}}$ and the definition of Galilean transformations, be the same.

- By the definition of Tr_*, $Tr_*(\mathsf{AxLine}^{\mathsf{STL}})$ is equivalent to

$$(\forall k, h \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{z} \in Q^4\big)(\forall e \in Ether)$$
$$\begin{bmatrix} (\exists \tau \in Q)\left(X_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = (\tau,0,0,0) \\ (\exists \tau \in Q)\left(X_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = (\tau,0,0,0) \\ (\exists \tau \in Q)\left(X_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{z}) = (\tau,0,0,0) \\ (\exists a \in Q)\big[\bar{z}-\bar{x} = a(\bar{y}-\bar{x}) \lor \bar{y}-\bar{z} = a(\bar{z}-\bar{x})\big] \end{bmatrix}$$

$X_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ X^{-1}_{\bar{v}_k(e)}$ is a linear bijection, so the images of $\bar{x}$, $\bar{y}$ and $\bar{z}$ are on a line if and only if $\bar{x}$, $\bar{y}$ and $\bar{z}$, are on a line. Since the images of $\bar{x}$, $\bar{y}$ and $\bar{z}$ are all on the time axis, this translation follows.

- By Lemma 11, $Tr_*(\mathsf{AxTriv}^{\mathsf{STL}})$ is equivalent to

$$(\forall T \in Triv)(\forall h \in IOb)(\exists k \in IOb)(\forall e \in Ether)$$
$$[X_{\bar{v}_k(e)} \circ w^{CK}_{hk} \circ X^{-1}_{\bar{v}_h(e)} = T].$$

To prove $Tr_*(\mathsf{AxTriv}^{\mathsf{STL}})$, we have to find an inertial observer k for every trivial transformation T and inertial observer h such that $X_{\bar{v}_k(e)} \circ w^{CK}_{hk} \circ X^{-1}_{\bar{v}_h(e)} = T$. Since T is an affine transformation it is a translation by a vector, say $\bar{z}'$, after a linear transformation, say T_0. By Lemma 12, if w^{CK}_{hk} is the translation by vector $\bar{z} = X^{-1}_{\bar{v}_k(e)}(\bar{z}')$ after T_0, then $X_{\bar{v}_k(e)} \circ w^{CK}_{hk} \circ X^{-1}_{\bar{v}_h(e)} = T$. By axiom $\mathsf{AxTriv}^{\mathsf{CK}}$, there is an inertial observer k such that w^{CK}_{hk} is the translation by vector $\bar{z} = X^{-1}_{\bar{v}_k(e)}(\bar{z}')$ after linear trivial transformation T_0. Hence there is an inertial observer k for which $X_{\bar{v}_k(e)} \circ w^{CK}_{hk} \circ X^{-1}_{\bar{v}_h(e)} = T$, and that is what we wanted to show.

- By Lemma 11, $Tr_*(\mathsf{AxAbsTime}^{\mathsf{STL}})$ is equivalent to

$$(\forall k, k' \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4\big)\big(\forall e \in Ether\big)$$
$$\left(\left[\begin{array}{l} \left(X_{\bar{v}_{k'}(e)} \circ w^{CK}_{kk'} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = \bar{x}' \\ \left(X_{\bar{v}_{k'}(e)} \circ w^{CK}_{kk'} \circ X^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = \bar{y}' \end{array}\right] \rightarrow time(\bar{x}, \bar{y}) = time(\bar{x}', \bar{y}')\right).$$

Since $X_{\bar{v}_{k'}(e)} \circ w^{CK}_{kk'} \circ X^{-1}_{\bar{v}_k(e)}$ is a Galilean transformation, it conserves absolute time. Therefore, the above follows from the definition of Galilean transformations and $\mathsf{AxAbsTime}^{\mathsf{CK}}$

- By Lemma 13, $Tr_*(\mathsf{AxEther}^{\mathsf{STL}})$ is equivalent to $\mathsf{AxEther}^{\mathsf{CK}}$.

- By the definition of $X_{\bar{v}_k(e)}$ and Lemma 13, $Tr_*(\mathsf{AxThExp}^{\mathsf{STL}})$ is equivalent to

$$(\exists h \in B)[IOb(h)]\wedge$$
$$(\forall e \in Ether)\big(\forall \bar{x}, \bar{y} \in Q^4\big)\Bigg(space(\bar{x}, \bar{y}) < \mathfrak{c}_{\mathfrak{e}} \cdot time(\bar{x}, \bar{y})$$
$$\rightarrow (\exists k \in IOb)\left[\begin{array}{l} (\exists \tau \in Q)\left(X_{\bar{v}_k(e)} \circ w^{CK}_{ek} \circ X^{-1}_{\bar{v}_e(e)}\right)(\bar{x}) = (\tau, 0, 0, 0) \\ (\exists \tau \in Q)\left(X_{\bar{v}_k(e)} \circ w^{CK}_{ek} \circ X^{-1}_{\bar{v}_e(e)}\right)(\bar{y}) = (\tau, 0, 0, 0) \end{array}\right]\Bigg),$$

which is, since $X^{-1}_{\bar{v}_e(e)} = X^{-1}_{\bar{0}} = Id$ is equivalent to

$$(\exists h \in B)[IOb(h)]\wedge$$
$$(\forall e \in Ether)\big(\forall \bar{x}, \bar{y} \in Q^4\big)\Bigg(space(\bar{x}, \bar{y}) < \mathfrak{c}_{\mathfrak{e}} \cdot time(\bar{x}, \bar{y})$$
$$\rightarrow (\exists k \in IOb)\left[\begin{array}{l} (\exists \tau \in Q)\left(X_{\bar{v}_k(e)} \circ w^{CK}_{ek}\right)(\bar{x}) = (\tau, 0, 0, 0) \\ (\exists \tau \in Q)\left(X_{\bar{v}_k(e)} \circ w^{CK}_{ek}\right)(\bar{y}) = (\tau, 0, 0, 0) \end{array}\right]\Bigg).$$

The first conjunct follows immediately as that is the same as the first conjunct of $\mathsf{AxThExp}^{\mathsf{CK}}_{+}$. If $x_0 = y_0$ then $time(\bar{x}, \bar{y}) = 0$, and $space(\bar{x}, \bar{y})$ cannot be smaller than zero, the antecedent $space(\bar{x}, \bar{y}) < \mathfrak{c}_{\mathfrak{e}} \cdot time(\bar{x}, \bar{y})$ is false and the implication is true. Let us now assume that $x_0 \neq y_0$. Let us denote the velocity corresponding to the line containing both $\bar{x}$ and $\bar{y}$ by $\bar{v}$. $|\bar{v}| < \mathfrak{c}_{\mathfrak{e}}$ as $space(\bar{x}, \bar{y}) < \mathfrak{c}_{\mathfrak{e}} \cdot time(\bar{x}, \bar{y})$. Worldview transformation w^{CK}_{ek} is $G^{-1}_{\bar{v}_k(e)} \circ M$ for a translation M by some spacetime vector. By the definition of $X_{\bar{v}_k(e)}$, if $\bar{v}_k(e) = \frac{\bar{v}}{\mathfrak{c}_{\mathfrak{e}} - \bar{v}}$, then $X_{\bar{v}_k(e)} \circ w^{CK}_{ek}$ is $G^{-1}_{\bar{v}} \circ M$ which always maps $\bar{x}$ and $\bar{y}$ to a line parallel to the time-axis; and with an appropriately chosen translation M, $G^{-1}_{\bar{v}} \circ M$ maps $\bar{x}$ and $\bar{y}$ to the time-axis. Therefore, by $\mathsf{AxTriv}^{\mathsf{CK}}$ and $\mathsf{AxThExp}^{\mathsf{CK}}_{+}$ there is an observer k such that $X_{\bar{v}_k(e)} \circ w^{CK}_{ek}$ maps $\bar{x}$ and $\bar{y}$ to the time-axis. So $Tr_*(\mathsf{AxThExp}^{\mathsf{STL}})$ follows from $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$.

- By Lemma 13 and Corollary 7, $Tr_*(\mathsf{AxNoFTL})$ is equivalent to

$$(\forall k \in IOb)(\exists e \in Ether)\big[Tr_*\big(speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big)\big].$$

Since $speed_e(k) = speed_k(e)$, this is equivalent to

$$(\forall k \in IOb)(\exists e \in Ether)\big[Tr_*\big(speed_k(e) < \mathfrak{c}_{\mathfrak{e}}\big)\big].$$

By Lemma 15, this is equivalent to

$$(\forall k \in IOb)(\forall e \in Ether)(\exists v \in Q^3)\left[\begin{array}{c} speed_k(e) = \frac{v}{\mathfrak{c}_{\mathfrak{e}} - v} \\ v < \mathfrak{c}_{\mathfrak{e}} \end{array}\right],$$

which is equivalent to

$$(\forall k \in IOb)(\forall e \in Ether)\left[\frac{\mathfrak{c}_{\mathfrak{e}} \cdot speed_k(e)}{1 + speed_k(e)} < \mathfrak{c}_{\mathfrak{e}}\right],$$

which is true since $speed_k(e) < 1 + speed_k(e)$ always holds because $\mathfrak{c}_{\mathfrak{e}}$ and $speed_k(e)$ are positive.

- $Tr_*(\mathsf{AxNoAcc}^{\mathsf{STL}})$ is equivalent to

$$(\forall k \in B)(\exists \bar{x} \in Q^4)(\exists b \in B)(\forall e \in Ether)$$
$$\left(\left[\begin{array}{c} b \notin IOb \to W^{CK}\big(k, b, X^{-1}_{\bar{v}_k(e)}(\bar{x})\big) \\ b \in IOb \to (\exists t \in Q)\big[w^{CK}_{kb}\big(X^{-1}_{\bar{v}_k(e)}(\bar{x})\big) = X^{-1}_{\bar{v}_b(e)}(t,0,0,0)\big] \end{array}\right] \to IOb(k)\right).$$

If k is an inertial observer, then the consequent of the implation is true, and hence the formula is true whatever the truth value of the antecedent. If k is not an inertial observer, it by $\mathsf{AxNoAcc}$ has no worldview and hence $v_k(e)$ is not defined, which makes the antecedent false. □

Lemma 17. Assume $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$. Let e be an ether observer, and let k and h be inertial. Assume that w^{CK}_{hk} is a trivial transformation which is translation by vector $\bar{z}$ after linear trivial transformation T. Then $Y_{\bar{v}_k(e)} \circ w^{SDL}_{hk} \circ Y^{-1}_{\bar{v}_h(e)}$ is the trivial transformation which is translation by vector $Y_{\bar{v}_k(e)}(\bar{z})$ after T.

Proof. The proof is analogous to the proof for Lemma 12. □

Theorem 10. Tr'_* is an interpretation of $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$ to $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$, i.e.,

$$\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr'_*(\varphi) \quad \text{if} \quad \mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}} \vdash \varphi.$$

Proof.

- $Tr'_*(\mathsf{EField}^{\mathsf{CK}})$ follows from $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ because $Tr'_*(\mathsf{EField}^{\mathsf{CK}}) \equiv \mathsf{EField}^{\mathsf{STL}} \in \mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$.

- $Tr'_*(\mathsf{AxSelf}^{\mathsf{CK}})$ is equivalent to the following formula:

$$(\forall k \in IOb)(\forall t, x, y, z \in Q)\Big((\forall e \in Ether)$$
$$(\exists \tau \in Q)\big[Y_{\bar{v}_k(e)}\big(w^{STL}_{kk}\big(Y^{-1}_{\bar{v}_k(e)}(\bar{x})\big)\big) = (\tau, 0, 0, 0)\big] \leftrightarrow x = y = z = 0\Big).$$

  Since $w^{STL}_{kk} = Id$, $Tr_*(\mathsf{AxSelf}^{\mathsf{CK}})$ is equivalent to:

$$(\forall k \in IOb)(\forall t, x, y, z \in Q)\Big((\forall e \in Ether)$$
$$(\exists \tau \in Q)\big[\bar{x} = (\tau, 0, 0, 0)\big] \leftrightarrow x = y = z = 0\Big)$$

  which is a tautology since $(t, x, y, z) = (\tau, 0, 0, 0)$ for some $\tau \in Q$ exactly if $x = y = z = 0$.

- $Tr'_*(\mathsf{AxEv}^{\mathsf{CK}})$ is equivalent to the following formula:

$$(\forall k, h \in IOb)\big(\forall \bar{x} \in Q^4\big)\big(\exists \bar{y} \in Q^4\big)(\forall e \in Ether)$$
$$\left[\left(Y_{\bar{v}_h(e)} \circ w^{STL}_{kh} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = \bar{y}\right],$$

  which follows from $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ since $Y_{\bar{v}_{k'}(e)} \circ w^{STL}_{kk'} \circ Y^{-1}_{\bar{v}_k(e)}$ is a Galilean transformation.

- $Tr'_*(\mathsf{AxSymD}^{\mathsf{CK}})$ is equivalent to the following formula:

$$(\forall k, k' \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4\big)(\forall e \in Ether)$$
$$\left(\left[\begin{array}{c} time(\bar{x}, \bar{y}) = time(\bar{x}', \bar{y}') = 0 \\ \left(Y_{\bar{v}_{k'}(e)} \circ w^{STL}_{kk'} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = \bar{x}' \\ \left(Y_{\bar{v}_{k'}(e)} \circ w^{STL}_{kk'} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = \bar{y}' \end{array}\right] \rightarrow space(\bar{x}, \bar{y}) = space(\bar{x}', \bar{y}')\right).$$

Since $\bar{x}$ and $\bar{y}$ are simultaneous before and after the Galilean transformation $Y_{\bar{v}_{k'}(e)} \circ w^{STL}_{kk'} \circ Y^{-1}_{\bar{v}_k(e)}$, their distance before and after the transformation must, by $\mathsf{AxSymD^{STL}}$ and the definition of Galilean transformations, be the same.

- $Tr'_*(\mathsf{AxLine^{CK}})$ is equivalent to

$$(\forall k, h \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{z} \in Q^4\big)(\forall e \in Ether)$$
$$\begin{bmatrix} (\exists \tau \in Q)\left(Y_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = (\tau, 0, 0, 0) \\ (\exists \tau \in Q)\left(Y_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = (\tau, 0, 0, 0) \\ (\exists \tau \in Q)\left(Y_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{z}) = (\tau, 0, 0, 0) \\ (\exists a \in Q)\big[\bar{z} - \bar{x} = a(\bar{y} - \bar{x}) \vee \bar{y} - \bar{z} = a(\bar{z} - \bar{x})\big] \end{bmatrix}$$

$Y_{\bar{v}_h(e)} \circ w^{CK}_{kh} \circ Y^{-1}_{\bar{v}_k(e)}$ is a linear bijection, so the images of $\bar{x}$, $\bar{y}$ and $\bar{z}$ are on a line if and only if $\bar{x}$, $\bar{y}$ and $\bar{z}$, are on a line. Since the images of $\bar{x}$, $\bar{y}$ and $\bar{z}$ are all on the time axis, this translation also follows.

- $Tr'_*(\mathsf{AxTriv^{CK}})$ is equivalent to

$$(\forall T \in Triv)(\forall h \in IOb)(\exists k \in IOb)(\forall e \in Ether)$$
$$[X_{\bar{v}_k(e)} \circ w^{STL}_{hk} \circ Y^{-1}_{\bar{v}_h(e)} = T].$$

To prove $Tr'_*(\mathsf{AxTriv^{CK}})$, we have to find an inertial observer k for every trivial transformation T and inertial observer h such that $Y_{\bar{v}_k(e)} \circ w^{STL}_{hk} \circ Y^{-1}_{\bar{v}_h(e)} = T$. Since T is an affine transformation it is a translation by a vector, say $\bar{z}'$, after a linear transformation, say T_0. By Lemma 17, if w^{STL}_{hk} is the translation by vector $\bar{z} = Y^{-1}_{\bar{v}_k(e)}(\bar{z}')$ after T_0, then $Y_{\bar{v}_k(e)} \circ w^{STL}_{hk} \circ Y^{-1}_{\bar{v}_h(e)} = T$. By axiom $\mathsf{AxTriv^{STL}}$, there is an inertial observer k such that w^{STL}_{hk} is the translation by vector $\bar{z} = Y^{-1}_{\bar{v}_k(e)}(\bar{z}')$ after linear trivial transformation T_0. Hence there is an inertial observer k for which $Y_{\bar{v}_k(e)} \circ w^{STL}_{hk} \circ Y^{-1}_{\bar{v}_h(e)} = T$, and that is what we wanted to show.

- $Tr'_*(\mathsf{AxAbsTime}^{CK})$ is equivalent to

$$(\forall k, k' \in IOb)\big(\forall \bar{x}, \bar{y}, \bar{x}', \bar{y}' \in Q^4\big)\big(\forall e \in Ether\big)$$
$$\left(\begin{bmatrix} \left(Y_{\bar{v}_{k'}(e)} \circ w^{STL}_{kk'} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = \bar{x}' \\ \left(Y_{\bar{v}_{k'}(e)} \circ w^{STL}_{kk'} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = \bar{y}' \end{bmatrix} \rightarrow time(\bar{x}, \bar{y}) = time(\bar{x}', \bar{y}') \right).$$

Since $Y_{\bar{v}_{k'}(e)} \circ w^{STL}_{kk'} \circ Y^{-1}_{\bar{v}_k(e)}$ is a Galilean transformation, it conserves absolute time. Therefore, the above follows from the definition of Galilean transformations and $\mathsf{AxAbsTime^{STL}}$.

- By Lemma 14, $Tr'_*(\mathsf{AxEther}^{CK})$ is $\mathsf{AxEther}^{STL}$.
- $Tr'_*(\mathsf{AxThExp}_+)$ is equivalent to

$$(\exists h \in B)\big[IOb(h)\big] \land (\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big)(\forall e \in Ether)$$
$$\left(x_0 \neq y_0 \to \big(\exists h \in IOb\big)\left[\begin{array}{c}(\exists \tau \in Q)\left(Y_{\bar{v}_h(e)} \circ w^{STL}_{kh} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{x}) = (\tau, 0, 0, 0)\\ (\exists \tau \in Q)\left(Y_{\bar{v}_h(e)} \circ w^{STL}_{kh} \circ Y^{-1}_{\bar{v}_k(e)}\right)(\bar{y}) = (\tau, 0, 0, 0)\end{array}\right]\right).$$

Since the worldviews of any two inertial observers differ only in a Galilean transformation it is enough to prove $Tr'_*(\mathsf{AxThExp}_+)$ when k is an ether observer. So it is enough to prove the following

$$(\exists h \in B)\big[IOb(h)\big] \land (\forall k \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big)$$
$$\left(x_0 \neq y_0 \to \big(\exists h \in Ether\big)\left[\begin{array}{c}(\exists \tau \in Q)\left(Y_{\bar{v}_h(e)} \circ w^{STL}_{eh}\right)(\bar{x}) = (\tau, 0, 0, 0)\\ (\exists \tau \in Q)\left(Y_{\bar{v}_h(e)} \circ w^{STL}_{eh}\right)(\bar{y}) = (\tau, 0, 0, 0)\end{array}\right]\right).$$

The first conjunct follows immediately as that is the same as the first conjunct of $\mathsf{AxThExp}^{\mathsf{STL}}$. If $x_0 = y_0$ the above statement is true as the antecedent of the implication is false. If $x_0 \neq y_0$, then let us denote the velocity corresponding to the line containing both $\bar{x}$ and $\bar{y}$ by $\bar{V}$. Worldview transformation w^{STL}_{ek} is $G^{-1}_{\bar{v}_k(e)} \circ M$ for a translation M by some spacetime vector. By the definition of $Y_{\bar{v}_k(e)}$, if $\bar{v}_k(e) = \frac{\mathfrak{c}_{\mathfrak{e}} \cdot \bar{V}}{1+\bar{V}}$, then $Y_{\bar{v}_k(e)} \circ w^{STL}_{ek}$ is $G^{-1}_{\bar{V}} \circ M$ which always maps $\bar{x}$ and $\bar{y}$ to a line parallel to the time-axis; and with an appropriately chosen translation M, $G^{-1}_{\bar{V}} \circ M$ maps $\bar{x}$ and $\bar{y}$ to the time-axis. Since $\bar{v}_k(e) = \frac{\mathfrak{c}_{\mathfrak{e}} \cdot \bar{V}}{1+\bar{V}} < \mathfrak{c}_{\mathfrak{e}}$, by $\mathsf{AxThExp}^{\mathsf{STL}}$ and $\mathsf{AxTriv}^{\mathsf{STL}}$ there is an observer k such that $Y_{\bar{v}_k(e)} \circ w^{STL}_{ek}$ maps $\bar{x}$ and $\bar{y}$ to the time-axis. So $Tr_*(\mathsf{AxThExp}^{\mathsf{STL}})$ follows from $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$.

- $Tr'_*(\mathsf{AxNoAcc}^{\mathsf{CK}})$ is equivalent to

$$(\forall k \in B)(\exists \bar{x} \in Q^4)(\exists b \in B)(\forall e \in Ether)$$
$$\left(\left[\begin{array}{c} b \notin IOb \to W^{STL}\big(k, b, Y^{-1}_{\bar{v}_k(e)}(\bar{x})\big)\\ b \in IOb \to (\exists t \in Q)\big[w^{STL}_{kb}\big(Y^{-1}_{\bar{v}_k(e)}(\bar{x})\big) = Y^{-1}_{\bar{v}_b(e)}(t, 0, 0, 0)\big]\end{array}\right] \to IOb(k)\right).$$

If k is an inertial observer, then the consequent of the implation is true, and hence the formula is true whatever the truth value of the antecedent. If k is not an inertial observer, it by $\mathsf{AxNoAcc}$ has no worldview and hence $v_k(e)$ is not defined, which makes the antecedent false. □

Theorem 11. Tr_* is a definitional equivalence between theories $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$ and $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$.

Proof. We only need to prove that the inverse translations of the translated statements are logical equivalent to the original statements since Tr_* and Tr'_* are interpretations by Theorem 9 and Theorem 10. Since Tr_* and Tr'_* are identical on every concept but the worldview relation, we only have to prove the following two statements:

$$\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}} \vdash Tr'_*\big(Tr_*[W^{STL}(k,b,\bar{x})]\big) \equiv W^{STL}(k,b,\bar{x}) \text{ and}$$
$$\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}} \vdash Tr_*\big(Tr'_*[W^{CK}(k,b,\bar{x})]\big) \equiv W^{CK}(k,b,\bar{x}).$$

The back and forth translation of W^{STL} is the following:

$$Tr'_*\big(Tr_*[W^{STL}(k,b,\bar{x})]\big) \equiv Tr'_*\Bigg((\forall e \in Ether)(\exists \bar{V} \in Q^3) \left[\begin{array}{c} \bar{V} = \bar{v}_b(e) \\ b \notin IOb \to W^{CK}\big(k,b,X^{-1}_{\bar{V}}(\bar{x})\big) \\ b \in IOb \to (\exists t \in Q)\big[\big(X_{\bar{V}} \circ w^{CK}_{kb} \circ X^{-1}_{\bar{V}}\big)(\bar{x}) = (t,0,0,0)\big] \end{array}\right]\Bigg),$$

which is equivalent to

$$(\forall e \in Ether)(\exists \bar{v}, \bar{V} \in Q^3) \left[\begin{array}{c} \bar{v} = \bar{v}_b(e) \wedge Tr'_*\big(\bar{v}_k(e) = \bar{V}\big) \\ b \notin IOb \to W^{STL}\Big(k,b,Y^{-1}_{\bar{v}}\big(X^{-1}_{\bar{V}}(\bar{x})\big)\Big) \\ b \in IOb \to (\exists t \in Q)\big[\big(X_{\bar{V}} \circ Y_{\bar{v}} \circ w^{STL}_{kb} \circ Y^{-1}_{\bar{v}} \circ X^{-1}_{\bar{V}}\big)(\bar{x}) = (t,0,0,0)\big] \end{array}\right].$$

By Lemma 9, this is equivalent to

$$(\forall e \in Ether) \left[\begin{array}{c} b \notin IOb \to W^{STL}\big(k,b,\bar{x}\big) \\ b \in IOb \to (\exists t \in Q)[w^{STL}_{kb}(\bar{x}) = (t,0,0,0)] \end{array}\right].$$

This is clearly equivalent to $W^{STL}(k,b,\bar{x})$ if b is not an inertial observer. If b is an inertial observer, then we need that $(\exists t \in Q)[w^{STL}_{kb}(\bar{x}) = (t,0,0,0)]$ is equivalent to $W^{STL}(k,b,\bar{x})$, which holds because of AxSelf in $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ and the definition of worldview transformation.

The proof $Tr_*\big(Tr'_*[W^{CK}(k,b,\bar{x})]\big) \equiv W^{CK}(k,b,\bar{x})$ from $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$ is completely analogous. □

Corollary 9. $Tr_* \circ Tr_+$ is a definitional equivalence between theories $\mathsf{SpecRel}^{\mathsf{e}}_{\mathsf{Full}}$ and $\mathsf{ClassicalKin}_{\mathsf{Full}}$.

Proof. By transitivity of definitional equivalence (Theorem 1), Theorem 8 and Theorem 11. □

10. CONCLUDING REMARKS

With the clarity of mathematical logic, we have achieved the following new results:

1. constructing an interpretation of special relativity into classical kinematics using Poincaré–Einstein synchronisation,
2. turning this interpretation into a definitional equivalence by extending special relativity with an ether concept and restricting classical kinematics to slower-than-light (STL) observers,
3. proving a definitional equivalence between classical kinematics and classical kinematics restricted to STL observers,
4. concluding by transitivity of definitional equivalence the main result that classical kinematics is definitional equivalent to special relativity extended with an ether concept.

To get special relativity theory, ether is the only concept that has to be removed from classical kinematics. However, removing ether from classical kinematics also leads to a change of the notions of space and time, which in the framework of this dissertation is handled by the translation functions.

It is philosophically interesting to note that our results are not identical to what we might expect from physical intuition: having or not having an upper speed limit appears to be an important physical distinction, but by Theorem 11 we can establish a definitional equivalence between theories with and without this speed limit. While we based translation Tr between $\mathsf{SpecRel}_{\mathsf{Full}}$ and $\mathsf{ClassicalKin}_{\mathsf{Full}}$ on the physical intuition of using a Poincaré–Einstein synchronisation, there was no such physical ground to translate between $\mathsf{ClassicalKin}^{\mathsf{STL}}_{\mathsf{Full}}$ and $\mathsf{ClassicalKin}^{\mathsf{NoAcc}}_{\mathsf{Full}}$.

Our results are all on kinematics; the similarities and differences between dynamic theories such as classical dynamics (Newton's Laws), Newton-Cartan theory (a geometrodynamical version of classical dynamics, which means that classical gravitation forces are expressed as curvatures of space and time like in relativity theory) and general relativity theory are good candidates for further research in the same spirit.

11. APPENDIX: SIMPLIFICATION OF TRANSLATED FORMULAS

The simplification tools in this appendix are (up to Lemma 22) consequences of all ether observers being at rest relative to each other in $\mathsf{ClassicalKin_{Full}}$.

Definition 31. Let b, k_1, …, k_n be variables of sort B. We say that formula φ is *ether-observer-independent* in variable b provided that k_1, …, k_n are inertial observers if the truth or falsehood of φ does not depend on to which ether we evaluated b if k_1, …, k_n are evaluated to inertial observers, that is:

$$EOI_b^{k_1,\ldots,k_n}[\varphi] \stackrel{\text{def}}{\Longleftrightarrow}$$
$$\mathsf{ClassicalKin_{Full}} \vdash (\forall k_1,\ldots,k_n \in IOb)(\forall e,e' \in Ether)[\varphi(e/b) \leftrightarrow \varphi(e'/b)].$$

where $\varphi(k/b)$ means that variable b gets replaced by k in all free occurrences of b in formula φ.

Let us note that the fewer variables are in the upper index of $EOI_b^{k_1,\ldots,k_n}(\varphi)$, the stronger statement we have about the ether-observer independence of formula φ. The strongest statement about the ether-observer independence of φ is $EOI_b(\varphi)$.

Lemma 18. Assuming $\mathsf{ClassicalKin_{Full}}$, if b is not a free variable of φ, then $EOI_b[\varphi]$ holds.

Proof. In the definition of EOI, there is nothing to replace in formula φ, so both parts $\varphi(e/b)$ and $\varphi(e'/b)$ of the equivalence remain the same. □

From Lemma 18, the following immediatly follows:

Corollary 10. Assume $\mathsf{ClassicalKin_{Full}}$. Let α and β be quantity terms and let k and h be body variables. Then atomic formulas $\alpha = \beta$ and $\alpha < \beta$ are ether-observer-independent, i.e., for any body variable b we have:

1. $EOI_b[\alpha = \beta]$.
2. $EOI_b[\alpha < \beta]$.
3. $EOI_b[k = k]$.

4. $EOI_b[k = h]$.

Let us note here that $EOI_b[b = b]$ holds and that $EOI_b[b = h]$ does not hold.

Lemma 19. Assume $\mathsf{ClassicalKin}_{\mathsf{Full}}$. Let k be a body variable. Then k being an inertial-observer or a light signal are ether-observer-independent, i.e., for any body variable b we have:

1. $EOI_b[IOb(b)]$.
2. $EOI_b[IOb(k)]$.
3. $EOI_b[Ph(b)]$.
4. $EOI_b[Ph(k)]$.

Proof. For the first item, note that all ether observers are inertial observers, by definition it means that $\mathsf{ClassicalKin}_{\mathsf{Full}} \vdash (\forall k_1, \ldots, k_n \in IOb)(\forall e, e' \in Ether)[IOb(e/b) \leftrightarrow IOb(e'/b)]$, which is true. $EOI_b[Ph(b)]$ is true because ether observers go slower-than-light, therefore both sides of the equivalence are false, which makes the equivalence true. The other two items follow from Lemma 18. □

Corollary 11. Assuming $\mathsf{ClassicalKin}_{\mathsf{Full}}$, the speed of an inertial observer is the same according to every ether observers, i.e.,

$$EOI_b^k[speed_b(k) = v].$$

Let us note that $EOI_b^k[\bar{v}_b(k) = \bar{v}]$ does not hold, because ether observers can be rotated relative to each other; hence the direction of $\bar{v}_b(k)$ depends on which ether observer variable b is evaluated to.

Corollary 12. Assuming $\mathsf{ClassicalKin}_{\mathsf{Full}}$, the velocity of all ether observers is the same according to every inertial observers, i.e.,

$$EOI_b^k[\bar{v}_k(b) = \bar{v}].$$

Corollary 13. Assuming $\mathsf{ClassicalKin}_{\mathsf{Full}}$, an inertial observer being slower than the speed of light is an ether-observer-independent statement, i.e.,

$$EOI_b^k[speed_b(k) < \mathfrak{c}_{\mathfrak{e}}].$$

The next three lemmas are being used to simplify the formulas that translation Tr provides us.

Lemma 20. Assuming $\mathsf{ClassicalKin}_{\mathsf{Full}}$ the following rules can be used to show the ether independence of complex formulas:

1. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[\neg\varphi]$.
2. If $*$ is a logical connective, then from $EOI_b^{k_1,\ldots,k_n}[\varphi]$ and $EOI_b^{h_1,\ldots,h_m}[\psi]$ follows $EOI_b^{k_1,\ldots,k_n,h_1,\ldots,h_m}[\varphi * \psi]$.
3. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[(\exists x \in Q)(\varphi)]$ and $EOI_b^{k_1,\ldots,k_n}[(\exists h \in B)(\varphi)]$.
4. From $EOI_b^{k_1,\ldots,k_n}[\varphi]$ follows $EOI_b^{k_1,\ldots,k_n}[(\forall x \in Q)(\varphi)]$ and $EOI_b^{k_1,\ldots,k_n}[(\forall h \in B)(\varphi)]$.

Proof. If $EOI_b^{k_1,\ldots,k_n}[\varphi]$ holds, then by definition

$$\mathsf{ClassicalKin_{Full}} \vdash (\forall k_1,\ldots,k_n \in IOb)(\forall e,e' \in Ether)[\varphi(e/b) \leftrightarrow \varphi(e'/b)],$$

which is equivalent to

$$\mathsf{ClassicalKin_{Full}} \vdash (\forall k_1,\ldots,k_n \in IOb)(\forall e,e' \in Ether)[\neg\varphi(e/b) \leftrightarrow \neg\varphi(e'/b)]$$

because $A \leftrightarrow B$ is equivalent to $\neg A \leftrightarrow \neg B$ and therefore $EOI_b^{k_1,\ldots,k_n}[\neg\varphi]$ also holds.

Let us now prove item 2. Since all logical connectives can be constructed from negation and conjunction, we only need to prove the property of item 3 for conjunction, as we have already proven it for negation in item 1 above.

To do so, we have to prove that in any model of $\mathsf{ClassicalKin_{Full}}$, $(\varphi \wedge \psi)(e/b)$ holds if and only if $(\varphi\wedge\psi)(e'/b)$ holds provided that e and e' are ether observers and $k_1,\ldots,k_n,h_1,\ldots,h_m$ are inertial observers. Formula $(\varphi \wedge \psi)(e/b)$ holds exactly if both $\varphi(e/b)$ and $\psi(e/b)$ hold. Similarly, $(\varphi\wedge\psi)(e'/b)$ holds exactly if both $\varphi(e'/b)$ and $\psi(e'/b)$ hold.

By $EOI_b^{k_1\ldots k_n}[\varphi]$, formula $\varphi(e/b)$ holds exactly if $\varphi(e'/b)$ holds provided that e and e' are ether observers and $k_1,\ldots,k_n$, are inertial observers. By $EOI_b^{h_1,\ldots,h_m}[\psi]$, formula $\psi(e/b)$ holds exactly if $\psi(e'/b)$ holds provided that e and e' are ether observers and $h_1,\ldots,h_m$ are inertial observers. Therefore, $(\varphi \wedge \psi)(e/b)$ holds exactly if $(\varphi \wedge \psi)(e'/b)$ holds provided that e and e' are ether observers and $k_1,\ldots,k_n,h_1,\ldots,h_m$ are inertial observers; and this is what we wanted to prove.

Item 3 is true because from $A \leftrightarrow B$ follows that $(\exists uA) \leftrightarrow (\exists uB)$.

Item 4 follows from items 1 and 3, since the universal quantifier can be composed of the negation and the existential quantifier: $\forall u(\varphi)$ is equivalent to $\neg\exists u(\neg\varphi)$. □

Lemma 21. Assume $\mathsf{ClassicalKin_{Full}}$. Let k be a body variable let and $\bar\alpha$ and $\bar\beta$ quantity terms. Then for any body variable b we have:

1. $EOI_b^k[Rad^{-1}_{\bar v_k(b)}(\bar\alpha) = \bar\beta]$.

2. $EOI_b^k[W(k,h,Rad^{-1}_{\bar{v}_k(b)}(\bar{\alpha}))]$.

Proof. To prove item 1, we have to prove that

$$\big(\forall \bar{v} \in Q^3\big)\big[\bar{v}_k(b) = \bar{v} \to Rad^{-1}_{\bar{v}}(\bar{\alpha}) = \bar{\beta}\big]$$

is ether-observer-independent in variable b provided that k is an inertial observer. $EOI_b^k[\bar{v}_k(b) = \bar{v}]$ holds because of Corollary 12. $EOI_b[Rad^{-1}_{\bar{v}}(\bar{\alpha}) = \bar{\beta}]$ holds because of Lemma 18. From Items 2 and 4 of Lemma 20 follows what we want to prove.

To prove Item 2, we have to prove that $W(k,h,\bar{\beta}) \wedge Rad^{-1}_{\bar{v}_k(b)}(\bar{\alpha}) = \bar{\beta}$ is ether-observer-independent in variable b provided that k is an inertial observer. $EOI_b^k[Rad^{-1}_{\bar{v}_k(b)}(\bar{\alpha}) = \bar{\beta}]$ holds because of Item 1 of this lemma. $EOI_b[W(k,h,\bar{\beta})]$ holds because of Lemma 18. From Item 2 of Lemma 20 follows what we want to prove. □

Lemma 22. Assume $\mathsf{ClassicalKin_{Full}}$ and that $EOI_e^{k_1,\ldots,k_n}[\varphi]$ and $EOI_e^{h_1,\ldots,h_m}[\psi]$ hold. For every logical connective $*$,

$$(\forall e \in Ether)(\varphi) * (\forall e \in Ether)(\psi)$$

is equivalent to

$$(\forall e \in Ether)(\varphi * \psi)$$

provided that $k_1,\ldots,k_n,h_1,\ldots,h_m$ are inertial observers.

Proof. From Lemma 20 Item 2, we know that $(\varphi*\psi)$ is ether-observer-independent in variable e provided that $k_1,\ldots,k_n,h_1,\ldots,h_m$ are inertial observers. Because φ, ψ and $\varphi*\psi$ are ether-observer-independent in variable e, it does not matter which ether observer we fill in there. Therefore, the two formulas are equivalent. □

Since our bounded quantifiers are just abbreviations, we can translate them in a straightforward way. To illustrate this, we will translate formulas $(\forall k \in IOb)[\varphi]$ and $(\exists k \in IOb)[\varphi]$.

$$\begin{aligned}
Tr\big((\forall k \in IOb^{SR})[\varphi]\big) &\equiv Tr\big(\forall k[IOb^{SR}(k) \to \varphi]\big) \equiv \\
&\forall k\Big(\big(IOb^{CK}(k) \wedge (\forall e \in Ether)\big[speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big]\big) \to Tr(\varphi)\Big) \equiv \\
&\forall k\Big(IOb^{CK}(k) \to \big((\forall e \in Ether)\big[speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big] \to Tr(\varphi)\big)\Big) \equiv \\
&(\forall k \in IOb^{CK})\Big((\forall e \in Ether)\big[speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big] \to Tr(\varphi)\Big).
\end{aligned}$$

$$\begin{aligned}
Tr\big((\exists k \in IOb^{SR})[\varphi]\big) &\equiv Tr\big(\exists k[IOb^{SR}(k) \wedge \varphi]\big) \equiv \\
&(\exists k \in IOb^{CK})\big((\forall e \in Ether)\big[speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big] \wedge Tr(\varphi)\big).
\end{aligned}$$

As an example on how we use this, the mechanographical translation of $\mathsf{AxSelf}^{\mathsf{SR}}$ is

$$\forall k\Big(IOb(k) \wedge \big((\forall e \in Ether)\big(speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big)$$
$$\rightarrow \big(\forall \bar{y} \in Q^4\big)(\forall e \in Ether)\big[W\big((k, k, Rad^{-1}_{\bar{v}_k(e)}(\bar{y})\big) \leftrightarrow y_1 = y_2 = y_3 = 0\big]\Big),$$

which by Lemma 22 and the bounded quantifiers notation can be simplified to

$$(\forall k \in IOb)(\forall e \in Ether)\Big(speed_e(k) < \mathfrak{c}_{\mathfrak{e}}$$
$$\rightarrow \big(\forall \bar{y} \in Q^4\big)\big[W\big((k, k, Rad^{-1}_{\bar{v}_k(e)}(\bar{y})\big) \leftrightarrow y_1 = y_2 = y_3 = 0\big]\Big).$$

Since the observers in the set E of "primitive ether" observers as introduced in the chapter on definitional equivalence are, by definition, only differing from each other by a trivial transformation, they are at rest relative to each other. So we can introduce a concept of "primitive ether independent observers" (PEIO) with the same properties as "ether independent observers" we have proven above. The proofs of these properties are analogous to the proofs for EIO in this appendix, which we for brevity will not repeat.

12. APPENDIX: OVERVIEW OF NOTATIONS

Sort	Natural language interpretation
B	set of bodies
Q	set of quantities

Language element	Natural language interpretation
$IOb(k)$	k is an inertial observer
$Ph(b)$	b is a light signal
$W(k, b, \bar{x})$	observer k coordinatizes body b at spatiotemporal coordinates $\bar{x}$
$+$	addition
$\cdot$	multiplication
$\leq$	less than or equal

We use the superscripts CK and SR to distinguish between the language of classical kinematics and special relativity theory.

Variable or term	Natural language interpretation
x, y, z, t, v, c	quantity variables (elements of Q)
α, β, γ	quantity terms
b, k, h, e, p	body variables (elements of B)
φ, ψ	logical statements
$\bar{x}, \bar{y}, \bar{z}$	spatiotemporal coordinates (elements of Q^4)

Defined symbols	Natural language interpretation
$wl_k(b)$	worldline of body b according to observer k
$ev_k(\bar{x})$	event occuring for observer k at coordinate point $\bar{x}$
$w_{kl}(\bar{x}, \bar{y})$	worldview transformation between observers k and l from $\bar{x}$ to $\bar{y}$
$Ether$	set of ether observers in classical kinematics
E	set of primitive ether observers

BIBLIOGRAPHY

[1] H. Andréka, J. X. Madarász, and I. Németi, with contributions from: A. Andai, G. Sági, I. Sain, and Cs. Tőke. *On the logical structure of relativity theories.* Research report, Alfréd Rényi Institute of Mathematics, Hungar. Acad. Sci., Budapest, http://www.math-inst.hu/pub/algebraic-logic/Contents.html, 2002.

[2] H. Andréka, J. X. Madarász, and I. Németi. *Mutual definability does not imply definitional equivalence, a simple example.* Mathematical Logic Quarterly 51,6; 591-597, 2005.

[3] H. Andréka, J. X. Madarász, and I. Németi. *Logical axiomatizations of space-time. Samples from the literature.* In A. Prékopa and E. Molnár, editors, *Non-Euclidean geometries*, pages 155–185. Springer-Verlag, New York, 2006.

[4] H. Andréka, J. X. Madarász, and I. Németi. *Logic of space-time and relativity theory.* In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of spatial logics*, pages 607–711. Springer-Verlag, Dordrecht, , 2007.

[5] H. Andréka, J. X. Madarász, I. Németi, and G. Székely. *A logic road from special relativity to general relativity.* Synthese, 186: 633 doi:10.1007/s11229-011-9914-8, 2012.

[6] H. Andréka, J. X. Madarász, I. Németi, and G. Székely. *What are the numbers in which spacetime, 2011.* in preparation.

[7] H. Andréka and I. Németi. *Comparing theories: the dynamics of changing vocabulary. A case-study in relativity theory.* In: Johan V. A. K. van Benthem on logical and informational dynamics. Editors: A. Baltag and S. Smets., Springer Series Outstanding contributions to logic Vol 5, Springer Verlag, pp.143-172, 2014.

[8] H. Andréka and I. Németi. *Definability Course Notes.* Department of Logic, ELTE, Budapest, spring 2014.

[9] J. B. Ax. *The elementary foundations of spacetime.* Found Phys 8(7/8):507-546, 1978.

[10] T. W. Barrett *Spacetime Structure.* Studies in History and Philosophy of Modern Physics. 51: 37-43, 2015.

[11] T. W. Barrett, H. Halvorson *Glymour and Quine on Theoretical Equivalence.* Forthcoming in The Journal of Philosophical Logic.

[12] T. W Barrett, H. Halvorson *Morita Equivalence.* Forthcoming in The Review of Symbolic Logic.

[13] J. L. Bell and M. Machover. *A course in Mathematical Logic.* North-Holland Publishing Company, Amsterdam, 1977.

[14] T. Benda. *An axiomatic foundation of relativistic spacetime.* Synthese 192 (7):1-16, 2013.

[15] M. Berger. *Geometry I.* Universitext. Springer-Verlag, Berlin. Translated from the 1977 French original by M. Cole and S. Levy, Fourth printing of the 1987 English translation, 2009.

[16] K. L. de Bouvère. *Logical Synonymity.* (Communicated by Prof. A. Heyting at the meeting of April 24, 1965, Mathematical Logic.

[17] C. C. Chang and H. J. Keisler. *Model theory.* North-Holland Publishing Co., Amsterdam, 1990.

[18] J. Corcoran, W. Frank and M. Maloney. *String Theory.* The Journal of Symbolic Logic, Vol. 39, No. 4, December 1974.

[19] J. Corcoran. *Notes and queries on definitional equivalence and related topics.* History and Philosophy of Logic, 1 p.231-234, 1980.

[20] A. Einstein. Zur Elektrodynamik bewegter Körper. *Annalen der Physik*, 17:891–921, 1905.

[21] H. B. Enderton. *A mathematical introduction to logic.* Academic Press, New York, 1972.

[22] H. A. Friedman and A. Visser. *When bi-interpretability implies synonymy.* pre-print

[23] M. Friend. *On the Epistemological Significance of the Hungarian Project.* Synthese, special edition: Logic and Relativity Theory. Vol. 192, Issue 7, pp. 2035-2051, 2015.

[24] R. Goldblatt. *Orthogonality and Spacetime Geometry.* Springer Verlag, 1987.

[25] A. K. Guts. *Axiomatic relativity theory.* Russian Mathematical Surveys Vol. 37 Nr. 2, 1982.

[26] L. Henkin, J. D. Monk and A. Tarski. *Cylindric Algebras, Part I.* North Holland, Amsterdam, 1971.

[27] W. Hodges. *Model Theory.* Cambridge University Press, 1993.

[28] L. Hudetz. *Definable Categorical Equivalence: Towards an Adequate Criterion of Theoretical Intertranslatability.* pre-print

[29] K. Lefever. *Logische en ontologische aspekten van ruimte en tijd in axiomatieken voor speciale relativiteitstheorie.* Master's thesis, Examencommissie van de Vlaamse Gemeenschap, Vrije Universiteit Brussel, Brussels, 2004.

[30] J. X. Madarász. *Logic and Relativity (in the light of definability theory).* PhD dissertation, Eötvös Loránd Univ., Budapest, 2002.

[31] J. X. Madarász and G. Székely. *Special Relativity over the Field of Rational Numbers* International Journal of Theoretical Physics 52(5):pp.1706-1718, 2013.

[32] A. A. Michelson and E. W. Morley. *On the Relative Motion of the Earth and the Luminiferous Ether.* American Journal of Science, 34, pp. 333-345, 1887.

[33] A. Molnár and G. Székely. *Axiomatizing Relativistic Dynamics using Formal Thought Experiments.* Synthese: 192:(7) pp. 2183-2222, 2015.

[34] R. Montague. *Contributions to the axiomatic foundations of set theory.* PhD dissertation, Berkeley, 1956.

[35] B. Mundy. *Optical Axiomatization of Minkowski Space-Time Geometry.* Philosophy of Science, Vol. 53, No. 1 March, 1986.

[36] V. Pambuccian. *Alexandrov-Zeeman type theorems expressed in terms of definability.* Aequationes Mathematicae, Vol. 74, pp. 249-261, 2007.

[37] J. H. Poincaré. *Œvres de Henri Poincaré 1842-1917.* Gauthier-Villars, Paris 1956.

[38] H. Reichenbach. *Axiomatik der relativistischen Raum-Zeit-Lehre.* , 1924.

[39] W. Rindler. *Relativity, Special, General and Cosmological.* Oxford University Press, Oxford, 2001.

[40] A. A. Robb. *Optical geometry of motion, a new view of the theory of relativity.* Cambridge, Heffner and Sons, 1911.

[41] J. W. Schutz. *Independent axioms for Minkowski space-time.* Pitman Research Notes in Mathematics Series Volume 373 , Addison-Wesley-Longman, Harlow, 1997.

[42] A. Sen, S. Bringsjord, N. Marton, J. Licato *Toward Diagrammatic Automated Discovery in Axiomatic Physics.* preliminary draft, presented at the 2nd International Conference on Logic, Relativity and Beyond, Budapest, 2015.

[43] M. Stannett and I. Németi *Using Isabelle/HOL to Verify First-Order Relativity Theory.* Journal of Automated Reasoning, Volume 52, Issue 4, pp 361378, April 2014.

[44] L. E. Szabó *Lorentzian Theories vs. Einsteinian Special Relativity A Logico-empiricist Reconstruction* Der Wiener Kreis in Ungarn, Volume 16 of the series Veröffentlichungen des Instituts Wiener Kreis pp 191-227

[45] G. Székely. *First-Order Logic Investigation of Relativity Theory with an Emphasis on Accelerated Observers.* PhD dissertation, Eötvös Loránd Univ., Budapest, 2009.

[46] G. Székely. *The existence of superluminal particles is consistent with the kinematics of Einstein's special theory of relativity.* Reports on Mathematical Physics No. 2 Vol. 72, 2013.

[47] G. Szekeres. *Kinematic geometry; an axiomatic system for Minkowski space-time.* Journal of the Australian Mathematical Society 8, 1968.

[48] A. Tarski and S. R. Givant. *A formalization of set theory without variables.* American Mathematical Society Colloquium Publicqtions, American Mathematical Society Vol. 41, 1987.

[49] J. O. Weatherall. *Are Newtonian gravitation and geometrized Newtonian gravitation theoretically equivalent?* Erkenntnis 81(5), 2016.

[50] J. O. Weatherall and J. B. Manchak. *The Geometry of Conventionality.* Philosophy of Science, 81 pp. 233-247, April 2014.

[51] J. A. Winnie. *The Causal Theory of Space-time.* "Foundations of Space-Time Theories" by J. Earman, C. N. Glymour and J, J. Stache, Minnesota Studies in the Philosophy of Science, Volume VIII, 1977.

[52] E. C. Zeeman. *Causality Implies the Lorentz Group.* J. Math. Phys. 5, 490, 1964.