Interpretation of Special Relativity in the Language of Newtonian Kinematics
(Work in progress - intermediate results)

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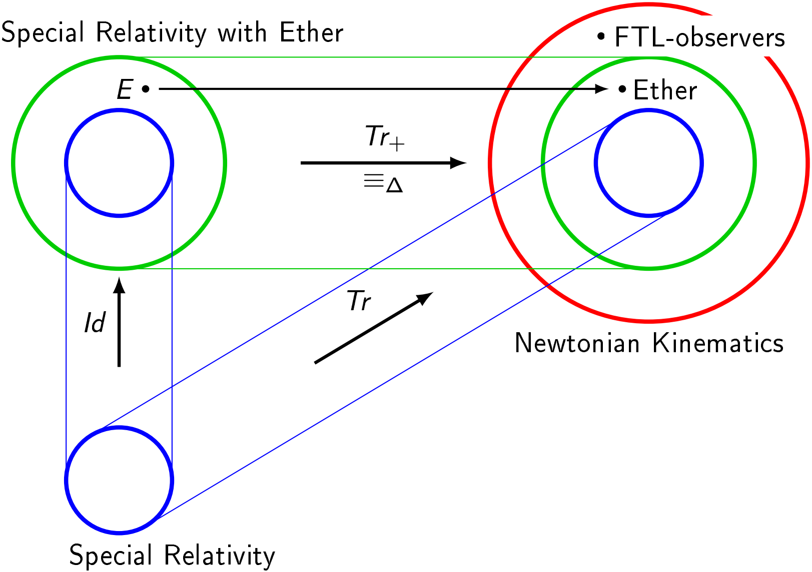
A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

An *interpretation* of theory $\mathbf{Th_1}$ in theory $\mathbf{Th_2}$ is a translation Tr which translates all axioms of $\mathbf{Th_1}$ into theorems of $\mathbf{Th_2}$.

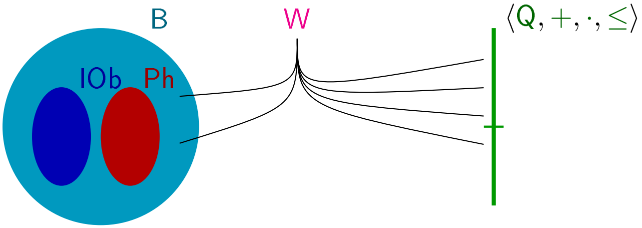
An interpretation is a *definitional equivalence* if a translated formula can be translated back such that it becomes a formula which is equivalent to the original formula.

There are translations Tr, Tr_+, and Tr_+^{-1} between the languages of NK and SR such that:

- $NK \vdash Tr(SR)$
- $NKnoFTL \vdash Tr_+(SRwithEther)$
- $SRwithEther \vdash Tr_+^{-1}(NKnoFTL)$
- Definitional equivalence: $SRwithEther \equiv_\Delta NKnoFTL$, i.e., Tr_+ and Tr_+^{-1} are inverses of each other up to logical equivalence in $NKnoFTL$ and $SRwithEther$.


Special Relativity with Ether
FTL-observers
E
Ether
Tr_+
$\equiv_\Delta$
Id
Tr
Newtonian Kinematics
Special Relativity

Language: $\{\, B, IOb, Ph, Q, +, \cdot, \leq, W \,\}$



B ⇝ Bodies (things that move)

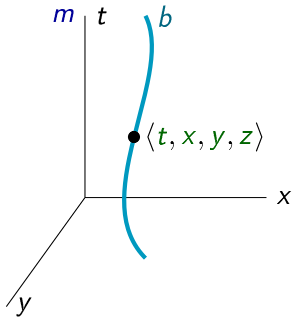
IOb ⇝ Inertial Observers Ph ⇝ Photons (light signals)

Q ⇝ Quantities

$+$, $\cdot$ and $\leq$ ⇝ field operations and ordering

W ⇝ Worldview (a 6-ary relation of type BBQQQQ)

$\mathsf{W}(m, b, t, x, y, z) \leftrightsquigarrow$ " observer m coordinatizes body b at spacetime location $\langle t, x, y, z\rangle$."



Worldline of body b according to observer m

$$wl_m(b) := \{\langle t, x, y, z\rangle \in \mathsf{Q}^4 \ : \ \mathsf{W}(m, b, t, x, y, z)\}$$

$$\textbf{Kin} := \{\mathsf{AxEField}, \mathsf{AxEv}, \mathsf{AxSelf}, \mathsf{AxSymD}, \mathsf{AxLine}, \mathsf{AxTriv}\}$$

$$\textbf{NewtonianKin}_F := \textbf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}_+^\uparrow\}$$

$$\textbf{SpecRel}_F := \textbf{Kin} \cup \{\mathsf{AxPh}_c, \mathsf{AxThExp}^\uparrow\}$$

AxEField :

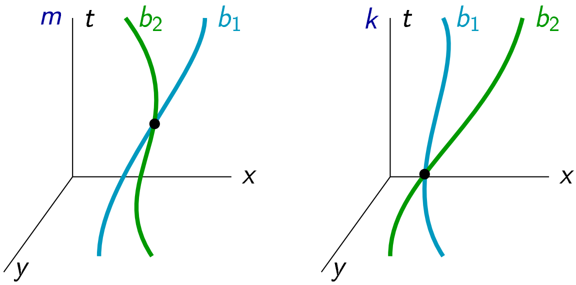
The **structure of quantities** $\langle \mathrm{Q}, +, \cdot, \leq \rangle$ *is an Euclidean field,*

- Real numbers: $\mathbb{R}$,
- Real algebraic numbers: $\overline{\mathbb{Q}} \cap \mathbb{R}$,
- Hyperreal numbers: $\mathbb{R}^*$,
- Real constructable numbers,
- Etc...

AxEv :

Inertial observers coordinatize the same events (meetings of bodies).

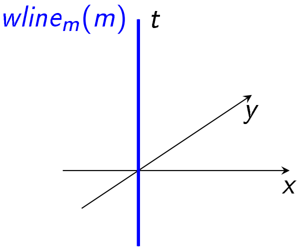
$$ev_m(\bar{x}) := \{b : W(m, b, \bar{x})\}$$



$$\forall m\, k\bar{x}\ \ [\mathsf{IOb}(m) \wedge \mathsf{IOb}(k) \rightarrow \exists \bar{y}\ \ ev_m(\bar{x}) = ev_k(\bar{y})].$$

AxSelf :

Every Inertial observer is stationary according to himself.

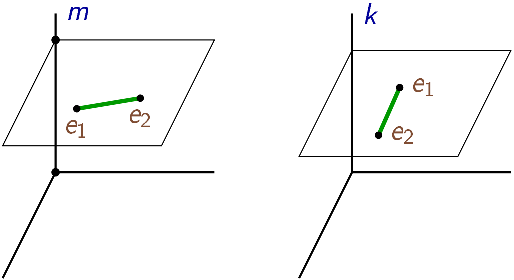


$$\forall mtxyz \ \Big(\mathsf{IOb}(m) \to \big[\mathsf{W}(m,m,t,x,y,z) \leftrightarrow x = y = z = 0\big]\Big).$$

AxSymD :

Inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them.

$$\mathrm{space}(\bar{x},\bar{y}) := \sqrt{(x_2 - y_2)^2 + \ldots + (x_d - y_d)^2}$$



$$\forall mk\bar{x}\bar{y}\bar{x}'\bar{y}' \;\; IOb(m) \wedge IOb(k) \wedge x_1 = y_1 \wedge x_1' = y_1' \wedge ev_m(\bar{x}) = ev_k(\bar{x}') \wedge ev_m(\bar{y}) = ev_k(\bar{y}') \rightarrow \mathrm{space}(\bar{x},\bar{y}) = \mathrm{space}(\bar{x}',\bar{y}')$$

AxLine :

The worldlines of inertial observers are straight lines according to inertial observers.

$$\forall m k \bar{x} \bar{y} \bar{z} \;\; IOb(m) \wedge IOb(k) \wedge W(m,k,\bar{x}) \wedge W(m,k,\bar{y}) \wedge W(m,k,\bar{z}) \\ \rightarrow \exists a \;\; \big(\bar{z} - \bar{x} = a(\bar{y} - \bar{x}) \vee \bar{y} - \bar{z} = a(\bar{z} - \bar{x})\big).$$

AxTriv :

Any trivial transformation of an inertial frame is also an inertial frame.

$(\forall T \in Triv)\forall m \exists k(w_{mk} = T)$, where Triv is the set of Trivial transformations, i.e. transformations that are isometries on space and translations on time.

AxAbsTime :

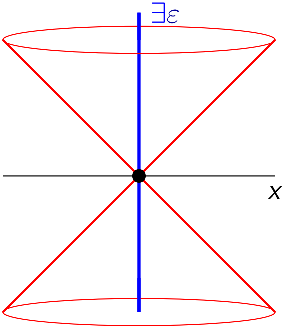
The time difference between two events is the same for all inertial observers.

$$\mathrm{time}(\bar{x},\bar{y}) := |x_1 - y_1|$$

$$\forall m k \bar{x}\bar{y}\bar{x}'\bar{y}' \;\; IOb(m) \wedge IOb(k) \wedge ev_m(\bar{x}) = ev_k(\bar{x}') \wedge ev_m(\bar{y}) = ev_k(\bar{y}') \rightarrow \mathrm{time}(\bar{x},\bar{y}) = \mathrm{time}(\bar{x}',\bar{y}').$$

AxEther(Einstein's AxLight) :

There exists an inertial observer in which the light cones are right.

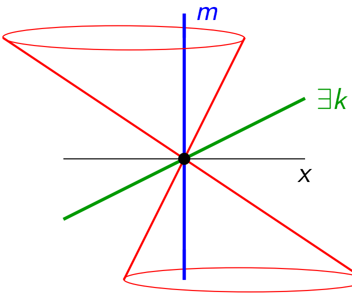


$$\exists \varepsilon c \Big[\mathsf{IOb}(\varepsilon) \land c > 0 \land \forall \bar{x}\bar{y} \Big(\exists p \big[\mathsf{Ph}(p) \land \mathsf{W}(\varepsilon, p, \bar{x}) \land \mathsf{W}(\varepsilon, p, \bar{y}) \big] \leftrightarrow \mathsf{space}(\bar{x}, \bar{y}) = c \cdot \mathsf{time}(\bar{x}, \bar{y}) \Big) \Big]$$

$\mathsf{AxThExp}^{\uparrow}_{+}$:

Inertial observers can move along any non-horizontal straight line.

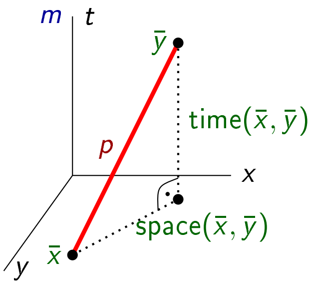
$$k \uparrow m \overset{def}{\iff} ev_k(\bar{x}) = ev_m(1,0,0,0) \land ev_k(\bar{y}) = ev_m(0,0,0,0) \rightarrow x_1 > y_1$$



$$\exists h \;\; IOb(h) \land \forall m\bar{x}\bar{y} \;\; \big(IOb(m) \land x_1 \neq y_1 \rightarrow \exists k \;\; IOb(k) \land W(m,k,\bar{x}) \land W(m,k,\bar{y}) \land m \uparrow k\big).$$

AxPh_c :

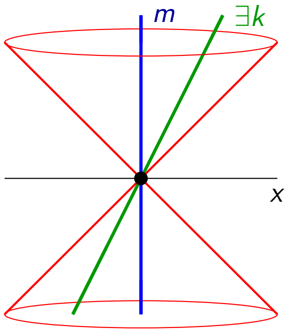
For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.



$$\exists c \Big[c > 0 \land \forall m \bar{x} \bar{y}\, \mathsf{IOb}(m) \rightarrow \Big(\exists p \big[\mathsf{Ph}(p) \land \mathsf{W}(m,p,\bar{x}) \land \mathsf{W}(m,p,\bar{y}) \big] \leftrightarrow \mathsf{space}(\bar{x},\bar{y}) = c \cdot \mathsf{time}(\bar{x},\bar{y}) \Big) \Big]$$

AxThExp$^{\uparrow}$:

Inertial observers can move with any *speed slower than that* of *light*.



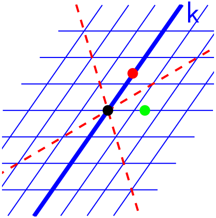
$$\exists h \ \ IOb(h) \land \forall m \bar{x} \bar{y} \ \ \big(IOb(m) \land \mathrm{space}(\bar{x}, \bar{y}) < c \cdot \mathrm{time}(\bar{x}, \bar{y}) \big) \rightarrow \exists k \ \ IOb(k) \land W(m, k, \bar{x}) \land W(m, k, \bar{y}) \land m \uparrow k \big).$$

$$\mathbf{Kin} := \{\mathsf{AxEField}, \mathsf{AxEv}, \mathsf{AxSelf}, \mathsf{AxSymD}, \mathsf{AxLine}, \mathsf{AxTriv}\}$$

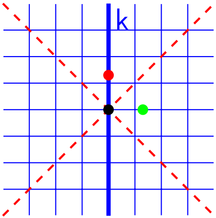
$$\mathbf{NewtonianKin}_F := \mathbf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}^{\uparrow}_{+}\}$$

$$\mathbf{SpecRel}_F := \mathbf{Kin} \cup \{\mathsf{AxPh}_c, \mathsf{AxThExp}^{\uparrow}\}$$

Galilean transformations:

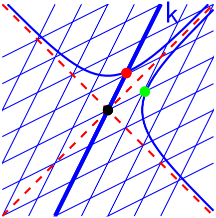


worldview of m

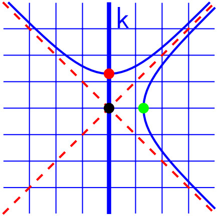


worldview of k

Poincaré transformations:

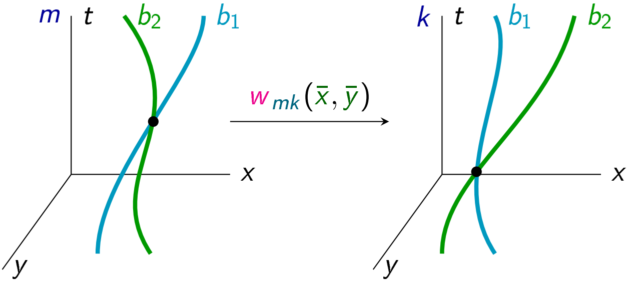


worldview of m



worldview of k

Worldview transformation :



$$w_{mk}(\bar{x}, \bar{y}) \stackrel{def}{\Longleftrightarrow} ev_m(\bar{x}) = ev_k(\bar{y})$$

Representation Theorems:

Theorem:

$SpecRel_F \vdash \forall(mk) IOb(m) \wedge IOb(k) \rightarrow$ "w_{mk} *is a Poincaré Transformation*".

- ca. 1998 proven for a version of BasAx strongly related to SpecRel in the "Big Book" by H. Andréka, J. X. Madarász & I. Németi
- Synthese 2012 "A logic road from special relativity to general relativity" by H. Andréka, J. X. Madarász, I. Németi & G. Székely (Theorem 2.2)

Theorem:

$NewtonianKin_F \vdash \forall(mk) IOb(m) \wedge IOb(k) \rightarrow$ "w_{mk} *is a Galilean Transformation*".

- proof for a different version of Newtonian Kinematics in the "Big Book" by H. Andréka, J. X. Madarász & I. Németi

Theorem:

$SpecRel_F \vdash \forall(kP) IOb(k) \wedge$ "P is a (orthochronous) Poincaré Transformation" $\rightarrow \exists k' w_{kk'} = P$.

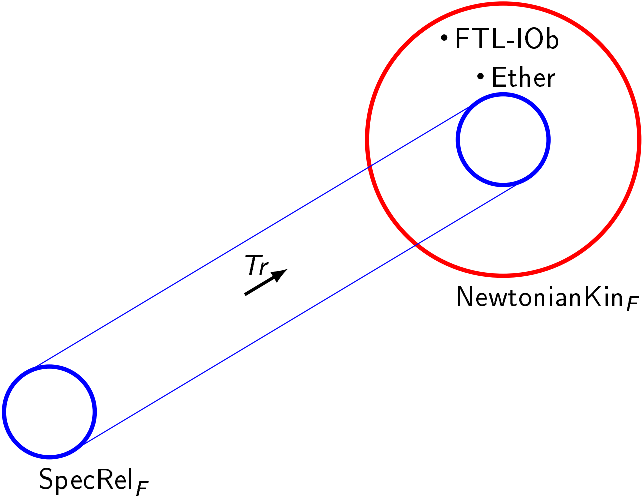
Theorem:

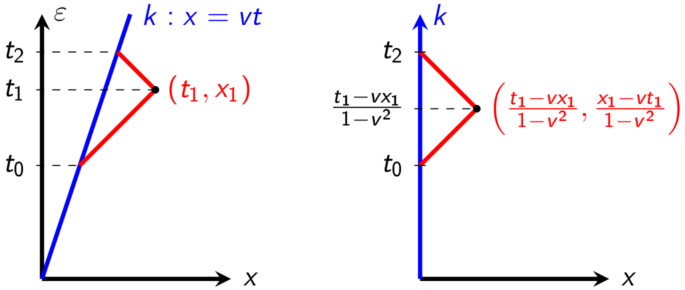
$NewtonianKin_F \vdash \forall(kG) IOb(k) \wedge$ "G is a (orthochronous) Galilean Transformation" $\rightarrow \exists k' w_{kk'} = G$.

- can be proven based on the ideas in the "Big Book" by H. Andréka, J. X. Madarász & I. Németi

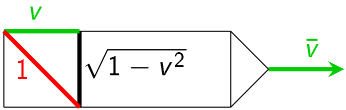
Theorem:

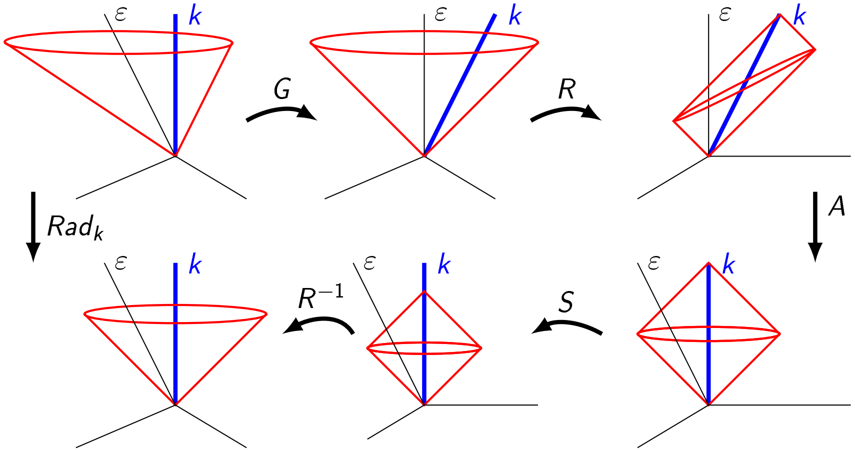
There is an interpretation Tr of **SpecRel_F** *in* **NewtonianKin_F**.





$$A = \begin{bmatrix} \frac{1}{1-v^2} & \frac{-v}{1-v^2} & 0 & 0 \\ \frac{-v}{1-v^2} & \frac{1}{1-v^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-v^2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{1-v^2}} \end{bmatrix} \qquad S = \sqrt{1 - v^2} \quad (\text{if } c = 1)$$





$$Rad_k = R^{-1} \circ S \circ A \circ R \circ G$$

$$Tr(a+b=c):=(a+b=c)$$

$$Tr(a\cdot b=c):=(a\cdot b=c)$$

$$Tr(a<b):=a<b$$

$$Tr\big(W_{SR}(k,b,t_r,x_r,y_r,z_r)\big):= \exists txyz[W_{NK}(k,b,t,x,y,z)\wedge Rad_k(t,x,y,z)=(t_r,x_r,y_r,z_r)]$$

$$Tr\big(IOb_{SR}(k)\big):=IOb_{NK}(k)\wedge\forall\varepsilon[Ether(\varepsilon)\rightarrow speed_{\varepsilon}^{NK}(k)<c]$$

where

$$Ether(\varepsilon)\overset{def}{\iff} IOb^{NK}(\varepsilon)\wedge\forall p[Ph(p)\rightarrow speed_{\varepsilon}^{NK}(p)=c]$$

Theorem:

There is an interpretation Tr of $\mathsf{SpecRel}_F$ *in* $\mathsf{NewtonianKin}_F$.

Proof:

- $\mathsf{NewtonianKin}_F \vdash \mathrm{Tr}(\mathsf{AxEField})$
- $\mathsf{NewtonianKin}_F \vdash \mathrm{Tr}(\mathsf{AxEv})$
- $\mathsf{NewtonianKin}_F \vdash \mathrm{Tr}(\mathsf{AxSelf})$
- $\mathsf{NewtonianKin}_F \vdash \mathrm{Tr}(\mathsf{AxSymD})$
- $\mathsf{NewtonianKin}_F \vdash \mathrm{Tr}(\mathsf{AxLine})$
- $\mathsf{NewtonianKin}_F \vdash \mathrm{Tr}(\mathsf{AxTriv})$
- $\mathsf{NewtonianKin}_F \vdash \mathrm{Tr}(\mathsf{AxPh}_c)$
- $\mathsf{NewtonianKin}_F \vdash \mathrm{Tr}(\mathsf{AxThExp}^{\uparrow})$

QED

$\mathsf{NewtonianKin}_F^{NoFTL} := \mathsf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}^{\uparrow}_{\mathsf{NoFTL}}, \mathsf{AxNoFTL}\}$

$\mathsf{AxThExp}^{\uparrow}_{\mathsf{NoFTL}}$:

Inertial observers can move with any speed which is in the ether frame slower than that of light.

$$\exists h \;\; IOb(h) \wedge \forall \varepsilon \bar{x} \bar{y} \Big(Ether(\varepsilon) \wedge \mathsf{space}(\bar{x}, \bar{y}) < c \cdot \mathsf{time}(\bar{x}, \bar{y}) \\ \rightarrow \exists k \;\; IOb(k) \wedge W(\varepsilon, k, \bar{x}) \wedge W(\varepsilon, k, \bar{y}) \wedge m \uparrow k \Big).$$

AxNoFTL :

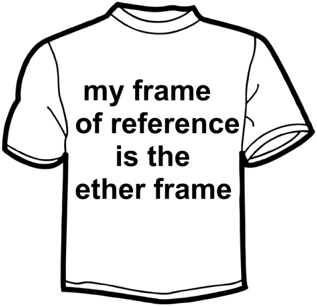
All inertial observers move slower than light with respect to the ether frames.

$$\forall m \varepsilon [IOb(m) \wedge Ether(\varepsilon) \rightarrow Speed^{NK}_{\varepsilon}(m) < c].$$

$\mathsf{SpecRel}^{\varepsilon}_{F} := \mathsf{SpecRel}_F \cup \{\text{AxPrimitiveEther}\}$

AxPrimitiveEther :

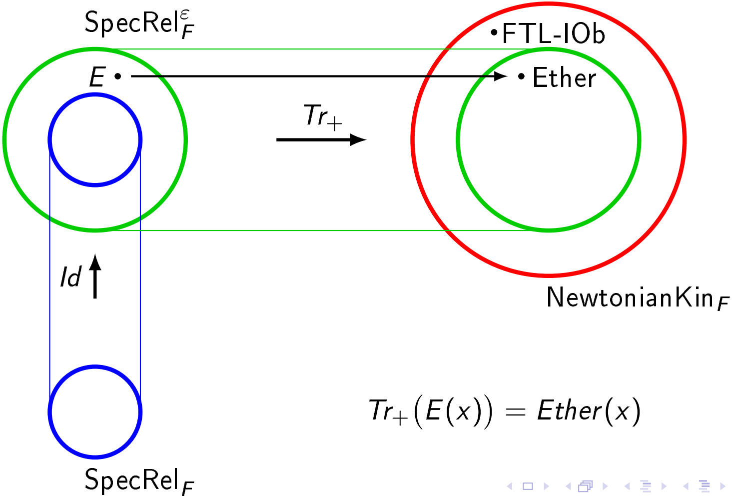
There is a non-empty class of ether observers, stationary with respect to each other, which is closed under trivial transformations.



$$\exists\varepsilon\big(E(\varepsilon)\wedge\forall k\big[[IOb(k)\wedge(\exists T\in Triv)w^{SR}_{\varepsilon k}=T]\leftrightarrow E(k)\big]\big)$$

Theorem:

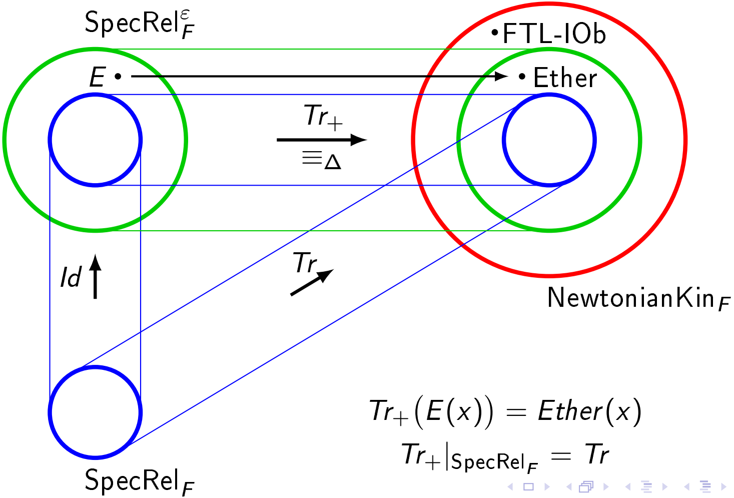
Tr_+ is a definitional equivalence between **$\mathsf{SpecRel}_F^\varepsilon$** *and* **$\mathsf{NewtonianKin}_F^{NoFTL}$**.



$$Tr_+(E(x)) = Ether(x)$$

Theorem:

Tr_+ is a definitional equivalence between **$\mathsf{SpecRel}_F^{\varepsilon}$** *and* **$\mathsf{NewtonianKin}_F^{NoFTL}$**.



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\mathsf{SpecRel}_F} = Tr$$