BSLPS YRD6
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Distances between Formal Theories
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Equivalence relations:

There are several ways in which theories can be equivalent to each other:

Logical Equivalence

$\Downarrow \not\Uparrow$

Definitional Equivalence

$\Downarrow \not\Uparrow$

Conceptual Equivalence (Morita Equivalence, definitional equivalence with different sorts)

$\Downarrow \not\Uparrow$

Categorical Equivalence

See:
"*Defining new universes in many-sorted logic*" by Hajnal Andréka, Judit X. Madarász and István Németi, Research Report, `https://www.researchgate.net/publication/242602426` (2008).
"*Morita Equivalence*" by Thomas William Barrett and Hans Halvorson, The Review of Symbolic Logic 9 (3): 556-582 (2016).

Non-equivalent formal theories:
Comparing scientific theories which are axiomatized in “First Order Logic with equality”.

- **Qualitative:**
  - Which concepts are similar or different between theories?
- **Quantitative:**
  - How many concepts are different?
  - How far are theories away from each other?

Definitions:

By a **cluster** (X, E) we mean a class X equipped with an equivalence relation E.

A **cluster network** is a triple $(\mathrm{X}, \mathrm{E}, \mathrm{S})$, where (X, E) is a cluster and S is a symmetric relation on X.

Given a cluster network $(\mathrm{X}, \mathrm{E}, \mathrm{S})$. A **path** leading from $T \in \mathrm{X}$ to $T' \in \mathrm{X}$ in $(\mathrm{X}, \mathrm{E}, \mathrm{S})$ is a finite sequence $b_1, \ldots, b_m$ of 0's and 1's such that there is a sequence $T_0, \ldots, T_m$ of members of X with $T_0 = T$, $T_m = T'$ and, for each $1 \leq i \leq m$,

$$b_i = 0 \iff T_{i-1} \, \mathrm{E} \, T_i \quad \text{and} \quad b_i = 1 \iff T_{i-1} \, \mathrm{S} \, T_i.$$

The **length** of this path is defined to be $\sum_{i=1}^{m} b_i$.

Two objects $T, T' \in \mathrm{X}$ are **connected** in $(\mathrm{X}, \mathrm{E}, \mathrm{S})$ iff there is a path leading from one of them to the other in $(\mathrm{X}, \mathrm{E}, \mathrm{S})$.

Let $\mathcal{X} = (\mathrm{X}, \mathrm{E}, \mathrm{S})$ be a cluster network. The **step distance on $\mathcal{X}$** is the function $\mathrm{d}_{\mathcal{X}} : \mathrm{X} \times \mathrm{X} \to \mathbb{N} \cup \{\infty\}$ defined as follows. For each $T, T' \in \mathrm{X}$:

- If T and T' are not connected in $(\mathrm{X}, \mathrm{E}, \mathrm{S})$, then $\mathrm{d}_{\mathcal{X}}(T, T') \stackrel{\text{def}}{\equiv} \infty$.
- If T and T' are connected in $(\mathrm{X}, \mathrm{E}, \mathrm{S})$, then $\mathrm{d}_{\mathcal{X}}(T, T') \stackrel{\text{def}}{\equiv} \min\{k \in \mathbb{N} : \exists$ a path leading from T to T' whose length is $k\}$.

Theorem: Step distance is a metric.

Let $\mathcal{X} = (\mathrm{X}, \mathrm{E}, \mathrm{S})$ be a cluster network and let $\mathrm{d}_{\mathcal{X}} : \mathrm{X} \times \mathrm{X} \to \mathbb{N} \cup \{\infty\}$ be the step distance on $\mathcal{X}$.
The following are true for each $T_1, T_2, T_3 \in \mathrm{X}$:

- $\mathrm{d}_{\mathcal{X}}(T_1, T_2) \geq 0$.
- $\mathrm{d}_{\mathcal{X}}(T_1, T_2) = 0 \iff T_1 \,\mathrm{E}\, T_2$.
- $\mathrm{d}_{\mathcal{X}}(T_1, T_2) = \mathrm{d}_{\mathcal{X}}(T_2, T_1)$.
- $\mathrm{d}_{\mathcal{X}}(T_1, T_2) \leq \mathrm{d}_{\mathcal{X}}(T_1, T_3) + \mathrm{d}_{\mathcal{X}}(T_3, T_2)$.

Trivial example: discrete distance

Let X be any class, let E be the identity relation and let $\mathrm{S} = \mathrm{X} \times \mathrm{X}$. Then, $\mathcal{X} = (\mathrm{X}, \mathrm{E}, \mathrm{S})$ is a cluster network and its step distance is the following discrete distance:

$$\mathrm{d}_{\mathcal{X}}(T, T') = \begin{cases} 0 & \text{if } T = T', \\ 1 & \text{if } T \neq T'. \end{cases}$$

Definition: Axiomatic distance

Suppose that we are given two theories T and T'.
We write $T \leftarrow T'$ iff there is $\varphi \in \mathrm{Fm}$ such that $T \cup \{\varphi\} \equiv T'$.
We also write $T - T'$ iff either $T \leftarrow T'$ or $T' \leftarrow T$.
We call the relation $\leftarrow$ **axiom adding**, while the converse relation $\rightarrow$ is called **axiom removal**.

Let X be a class of some theories and consider the cluster network $(\mathrm{X}, \equiv, -)$. We call the step distance Ad_{X} on this cluster network the **axiomatic distance** on X.

Theorem: Axiomatic distance is $0, 1, 2$, or infinite.

for all $T, T' \in \mathrm{X}$, we have the following

$$\mathrm{Ad}_{\mathrm{X}}(T, T') = \begin{cases} 0 & \text{if } T \equiv T', \\ 1 & \text{if } T' \text{ or } T \text{ is finitely axiomatizable over the other,} \\ \infty & \text{if } T \text{ and } T' \text{ are not connected in } (\mathrm{X}, \equiv, -), \\ 2 & \text{otherwise.} \end{cases}$$

Definitions:

Let T_1 and T_2 be two theories. We say that T_2 is a **conservative extension** of T_1, in symbols $T_1 \sqsubseteq T_2$, iff $\mathrm{Fm}_1 \subseteq \mathrm{Fm}_2$ and, for all $\varphi \in \mathrm{Fm}_1$, $T_2 \models \varphi \iff T_1 \models \varphi$.

A **concept** in theory T is a maximal set of logically equivalent formulas in T. In other words, a concept in T is the set $[\varphi]_T \stackrel{\text{def}}{\equiv} \{\psi \in \mathrm{Fm} : T \models \varphi \leftrightarrow \psi\}$, for some formula φ.

Definition: Conceptual distance

We say that theory T' is a **one-concept-extension** of theory T and we write $T \rightsquigarrow T'$ iff $\mathcal{L}' = \mathcal{L} \cup \{R\}$, for some relation symbol R, and $T \sqsubseteq T'$.
We also write $T \sim T'$ iff $T \rightsquigarrow T'$ or $T' \rightsquigarrow T$, and in this case we say that T and T' are **separated by at most one concept**.

Let X be a class of theories. The step distance Cd_{X} induced by the cluster network $(\mathrm{X}, \rightleftarrows, \sim)$ is called the **conceptual distance** on X.

Example: Using definitional equivalence to calculate the conceptual distance between classical kinematics and relativistic kinematics

See:

"*Using Logical Interpretation and Definitional Equivalence to Compare Classical Kinematics and Special Relativity Theory*" by Koen Lefever
PhD Dissertation, Vrije Universiteit Brussel, 2017-05-26.

"*Comparing Classical and Relativistic Kinematics in First-Order Logic*"
by Koen Lefever and Gergely Székely
Logique et Analyse, ISSN: 2295-5836, p. 57-117, Vol 61, Nr 241,
DOI: 10.2143/LEA.241.0.3275105, arXiv:1707.05371 (2018).

A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

An *interpretation* of theory T in theory T' is a translation Tr which translates all tautologies and all axioms of T into theorems of T'.

A *definitional equivalence* exists between two theories if those theories can be interpreted in each other and if all formulas from both theories translated into the other theory and back are logical equivalent to the original formulas.

Let SpecRel^{e} = SpecRel $\cup$ {AxPrimitiveEther}.

Axiom AxPrimitiveEther:

$$\exists v_0 \left[\mathsf{IOb}(v_0) \land \forall v_1 \left(E(v_1) \leftrightarrow [\mathsf{IOb}(v_1) \land \varphi(v_0, v_1)] \right) \right],$$

where IOb is a unary relation symbol represents inertial observers and $\varphi(v_0, v_1)$ is a formula in the language of SpecRel capturing that observers v_0 and v_1 are stationary with respect to each other. $\varphi(v_0, v_1)$ is a formula with two free variables in the language of SpecRel.

Theorem: $Cd(SpecRel, ClassicalKin) = 1$

There are translations Tr, Tr_+, and Tr'_+ between the languages of Classical Kinematics and Special Relativity Theory such that:

- $ClassicalKin \vdash Tr(SpecRel)$,
- $ClassicalKin^{STL} \vdash Tr_+(SpecRel^e)$,
- $SpecRel^e \vdash Tr'_+(ClassicalKin^{STL})$
- Definitional equivalence: $SpecRel^e \equiv_\Delta ClassicalKin^{STL}$, i.e., Tr_+ and Tr'_+ are inverses of each other up to logical equivalence in $SpecRel^e$ and $ClassicalKin^{STL}$.

And thus:

- $Cd(SpecRel, SpecRel^e) = 1$,
- $Cd(SpecRel^e, ClassicalKin^{STL}) = 0$,
- $Cd(SpecRel, ClassicalKin) \leq 2$.

There are translations Tr_*, and Tr'_* between the languages of Classical Kinematics and Classical Kinematics restricted to Slower-Than-Light oberservers such that:

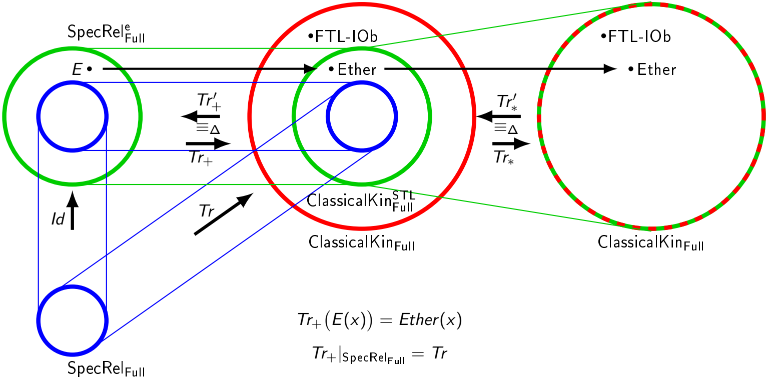
- $ClassicalKin^{STL} \vdash Tr_*(ClassicalKin)$,
- $ClassicalKin \vdash Tr'_*(ClassicalKin^{STL})$,
- Definitional equivalence: $ClassicalKin \equiv_\Delta ClassicalKin^{STL}$,

and hence by transitivity of definitional equivalence:

- $SpecRel^e \equiv_\Delta ClassicalKin$.

Thus:

- $Cd(ClassicalKin^{STL}, ClassicalKin) = 0$,
- $Cd(SpecRel, ClassicalKin) = 1$.



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{SpecRel_{Full}} = Tr$$

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