

Conceptual Distances between scientific theories

Joint work-in-progress

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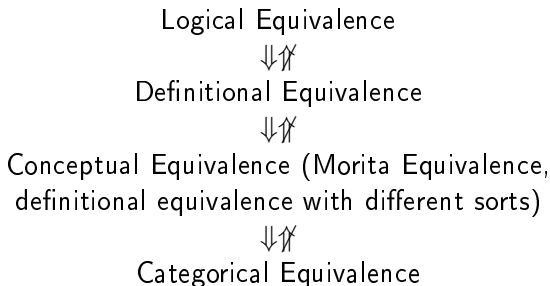
2017-11-27

Comparing scientific theories which are axiomatized in “First Order Logic with equality”:

- **Qualitative:**
 - Which concepts are similar or different between theories?
- **Quantitative:**
 - How many concepts are different?
 - How far are theories away from each other?

Equivalence relations:

There are several ways in which theories can be equivalent to each other:



See “*Morita Equivalence*” by Thomas William Barrett and Hans Halvorson,
The Review of Symbolic Logic 9 (3): 556-582 (2016).

Metric distance:

Let X be a set of theories and let \equiv be an equivalence relation on X . We call a map $d : X \times X \rightarrow [0, \infty)$ the conceptual distance on (X, \equiv) if it satisfies the following properties:

- 1 $d(x, y) \geq 0$,
- 2 $d(x, y) = 0$ iff $x \equiv y$,
- 3 $d(x, y) = d(y, x)$,
- 4 $d(x, z) \leq d(x, y) + d(y, z)$.

Conceptual distance: three approaches

- 1 A trivial example: discrete distance
- 2 An ad-hoc example: adding and removing concepts to make classical kinematics and special relativity equivalent
- 3 A definition using Cylindric Algebras or Concept Algebras

Trivial example: discrete distance

Let X be any set of theories and \equiv any equivalence relation on X . The discrete conceptual distance on (X, \equiv) is the following:

$$d(x, y) = \begin{cases} 0 & \text{if } x \equiv y, \\ 1 & \text{if } x \not\equiv y. \end{cases}$$

We define the *Conceptual Distance* between two theories as the minimum number of concepts (i.e. logical predicates) which need to be added to or subtracted from one theory to make it definitionally (or logically, categorically, etc. . .) equivalent to the other theory.

Two theories which are equivalent have a Conceptual Distance of zero.

Adding a concept:

- introduce new primitive predicate R
- add axiom $\exists \bar{x}(\varphi_R(\bar{x}))$

Removing concept $\varphi(\bar{x})$:

- add axiom $\neg \exists \bar{x}(\varphi(\bar{x}))$
- change theory T to a weaker theory T^- to avoid contradictions

Example: Using definitional equivalence to calculate the conceptual distance between classical kinematics and relativistic kinematics

See:

“Using Logical Interpretation and Definitional Equivalence to Compare Classical Kinematics and Special Relativity Theory” by Koen Lefever
PhD Dissertation, Vrije Universiteit Brussel, 2017-05-26

“Comparing Classical and Relativistic Kinematics in First-Order Logic” by Koen Lefever and Gergely Székely

Forthcoming in *Logique et Analyse*, 2018

A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

An *interpretation* of theory **Th₁** in theory **Th₂** is a translation Tr which translates all tautologies and all axioms of **Th₁** into theorems of **Th₂**.

A *definitional equivalence* exists between two theories if those theories can be interpreted in each other and if all formulas from both theories translated into the other theory and back are logical equivalent to the original formulas.

There are translations Tr , Tr_+ , and Tr'_+ between the languages of Classical Kinematics and Special Relativity Theory such that:

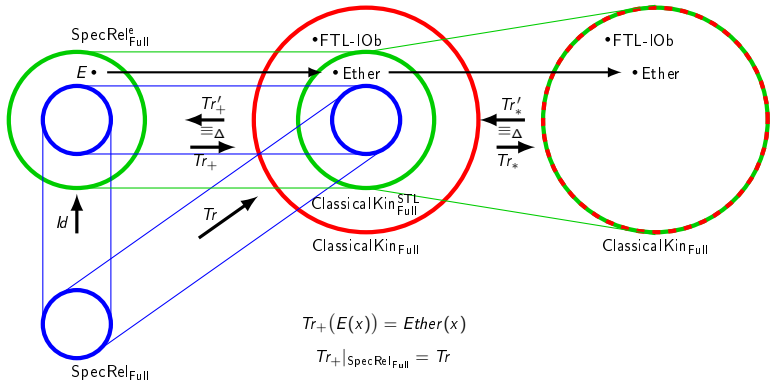
- $ClassicalKin_{Full} \vdash Tr(SpecRel_{Full})$
- $ClassicalKin_{Full}^{STL} \vdash Tr_+(SpecRel_{Full}^e)$
- $SpecRel_{Full}^e \vdash Tr'_+(ClassicalKin_{Full}^{STL})$
- Definitional equivalence: $SpecRel_{Full}^e \equiv_{\Delta} ClassicalKin_{Full}^{STL}$,
i.e., Tr_+ and Tr'_+ are inverses of each other up to logical
equivalence in $SpecRel_{Full}^e$ and $ClassicalKin_{Full}^{STL}$.

There are translations Tr_* , and Tr'_* between the languages of Classical Kinematics and Classical Kinematics restricted to Slower-Than-Light observers such that:

- $ClassicalKin_{Full}^{STL} \vdash Tr_*(ClassicalKin_{Full})$
- $ClassicalKin_{Full} \vdash Tr'_*(ClassicalKin_{Full}^{STL})$
- Definitional equivalence: $ClassicalKin_{Full} \equiv_{\Delta} ClassicalKin_{Full}^{STL}$,

and hence by transitivity of definitional equivalence:

- $SpecRel_{Full}^e \equiv_{\Delta} ClassicalKin_{Full}$.



$\mathbf{Kin} := \{\text{AxEField}, \text{AxEv}, \text{AxSelf}, \text{AxSymD}, \text{AxLine}, \text{AxTriv}, \text{AxNoAcc}\}$

$\mathbf{ClassicalKin}_{Full} := \mathbf{Kin} \cup \{\text{AxEther}, \text{AbsTime}, \text{AxThExp}_+\}$

$\mathbf{SpecRel}_{Full} := \mathbf{Kin} \cup \{\text{AxPh}_c, \text{AxThExp}\}$

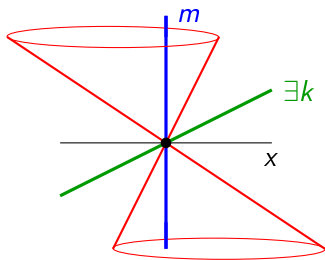
Kin:

- **AxEField**: The structure of quantities $\langle Q, +, \cdot, \leq \rangle$ is an Euclidean field,
- **AxEv**: Inertial observers coordinatize the same events (meetings of bodies).
- **AxSelf**: Every inertial observer is stationary according to himself.
- **AxSymD**: Inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them.
- **AxLine**: The worldlines of inertial observers are straight lines according to inertial observers.
- **AxTriv**: Any trivial transformation of an inertial observer is also an inertial observer.
- **AxNoAcc**: All observers are inertial observers.

ClassicalKin_{Full}:

- **AxAbsTime**: The time difference between two events is the same for all inertial observers.
- **AxEther**: There exists an inertial observer in which the light cones are right.
- **AxThExp₊**: Inertial observers can move along any non-horizontal straight line:

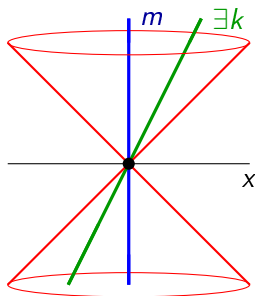
$$(\exists h \in B)[IOb(h)] \wedge (\forall k \in IOb)(\forall \bar{x}, \bar{y} \in Q^4) (x_0 \neq y_0 \rightarrow (\exists k' \in IOb)[\bar{x}, \bar{y} \in wl_k(k')]).$$



SpecRel_{Full}:

- **AxPh**: For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.
- **AxThExp**: Inertial observers can move with any speed slower than that of light:

$$(\exists h \in B)[IOb(h)] \wedge (\forall k \in IOb)(\forall \bar{x}, \bar{y} \in Q^4) \\ (space(\bar{x}, \bar{y}) < c \cdot time(\bar{x}, \bar{y}) \rightarrow (\exists k' \in IOb)[\bar{x}, \bar{y} \in wl_k(k')]).$$



$\mathbf{Kin} := \{AxEField, AxEv, AxSelf, AxSymD, AxLine, AxTriv, AxNoAcc\}$

$\mathbf{ClassicalKin}_{Full} := \mathbf{Kin} \cup \{AxEther, AbsTime, AxThExp_+\}$

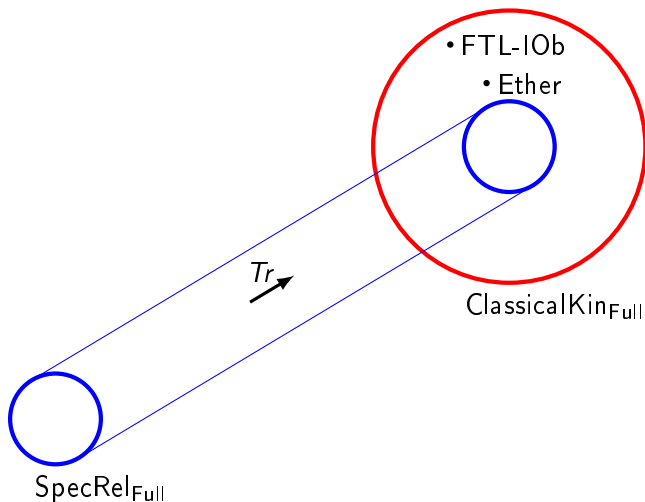
$\mathbf{SpecRel}_{Full} := \mathbf{Kin} \cup \{AxPh_c, AxThExp\}$

$\mathbf{ClassicalKin}_{Full} \vdash$ Worldview transformations are Galilean transformations.

$\mathbf{SpecRel}_{Full} \vdash$ Worldview transformations are Poincaré transformations (see “Big Book” by H. Andréka, J. X. Madarász, and I. Németi).

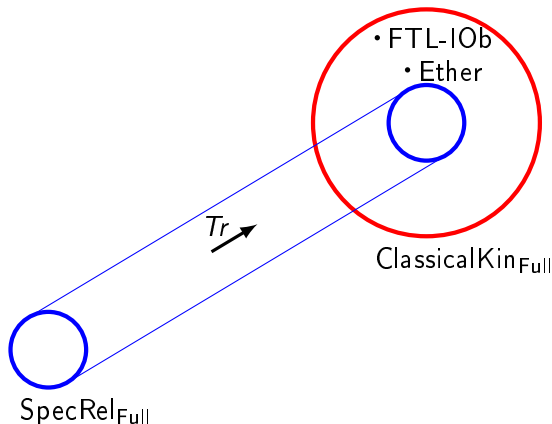
Theorem:

There is an interpretation Tr of SpecRel_{Full} in $\text{ClassicalKin}_{Full}$.



Theorem:

There is no definitional equivalence between SpecRel_{Full} and $\text{ClassicalKin}_{Full}$.



$$\text{Conceptual_distance}(\text{SpecRel}_{Full}, \text{ClassicalKin}_{Full}) > 0$$

Adding and removing concepts to make both theories equivalent:

- Removing FTL observers from **ClassicalKin_{Full}**
- Adding a “primitive ether” to **SpecRel_{Full}**

ClassicalKin^{STL}_{Full} :=
Kin \cup {AxEther, AbsTime, AxThExp^{STL}, AxNoFTL}

AxNoFTL :

All *inertial observers* move *slower than light* with respect to the ether frames.

$$\neg \exists m (IOb(m) \wedge \exists e [Ether(e) \wedge Speed_e^{CK}(m) \geq c_e]).$$

AxThExp^{STL} :

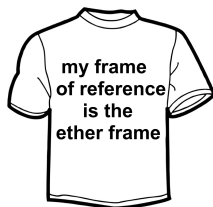
Inertial observers can move with any *speed* which is in the ether frame *slower than that of light*.

$$\begin{aligned} \exists h (IOb(h)) \wedge \forall e \bar{x} \bar{y} (Ether(e) \wedge \text{space}(\bar{x}, \bar{y}) < c_e \cdot \text{time}(\bar{x}, \bar{y})) \\ \rightarrow \exists k IOb(k) \wedge W(e, k, \bar{x}) \wedge W(e, k, \bar{y})). \end{aligned}$$

$$\text{SpecRel}_{Full}^e := \text{SpecRel}_{Full} \cup \{\text{AxPrimitiveEther}\}$$

AxPrimitiveEther :

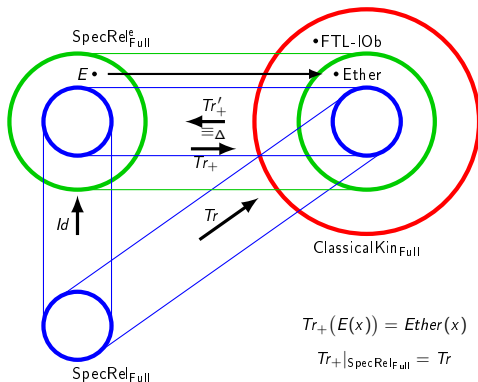
There is a non-empty class of ether observers, stationary with respect to each other, which is closed under trivial transformations.



$$\exists e (E(e) \wedge \forall k [[IOb(k) \wedge (\exists T \in Triv) w_{ek}^{SR} = T] \leftrightarrow E(k)])$$

Theorem:

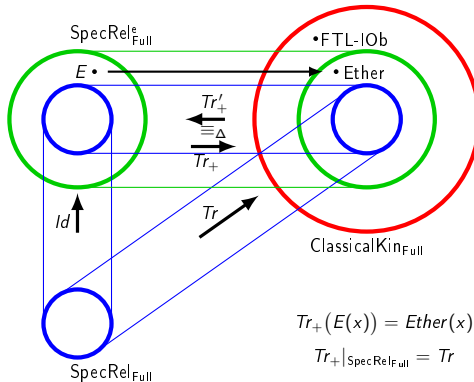
There is an interpretation Tr'_+ of $\text{ClassicalKin}_{Full}^{STL}$ in SpecRel_{Full}^e .

$$\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{ClassicalKin}_{Full}^{STL})$$


Theorem:

Tr_+ and Tr'_+ are a definitional equivalence between SpecRel_{Full}^e and $\text{ClassicalKin}_{Full}^{STL}$.

$$Tr'_+ \circ Tr_+(\text{SpecRel}_{Full}^e) \Leftrightarrow \text{SpecRel}_{Full}^e \quad \text{and} \quad Tr_+ \circ Tr'_+(\text{ClassicalKin}_{Full}^{STL}) \Leftrightarrow \text{ClassicalKin}_{Full}^{STL}$$



$$\text{Conceptual_distance}(\text{SpecRel}_{Full}, \text{ClassicalKin}_{Full}) \leq 2$$

Theorem:

There is an interpretation Tr_* of $\text{ClassicalKin}_{Full}^{STL}$ in $\text{ClassicalKin}_{Full}$.

$$\text{ClassicalKin}_{Full} \vdash Tr_*(\text{ClassicalKin}_{Full}^{STL})$$

Theorem:

There is an interpretation Tr'_* of $\text{ClassicalKin}_{Full}$ in $\text{ClassicalKin}_{Full}^{STL}$.

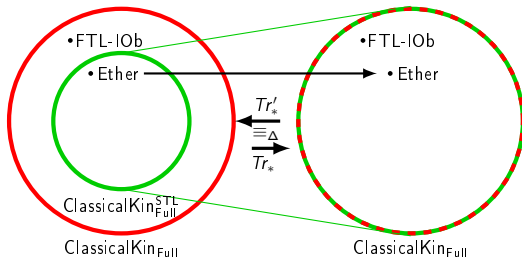
$$\text{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(\text{ClassicalKin}_{Full})$$

Theorem:

Tr_* and Tr'_* are a definitional equivalence between $\text{ClassicalKin}_{Full}^{STL}$ and $\text{ClassicalKin}_{Full}$.

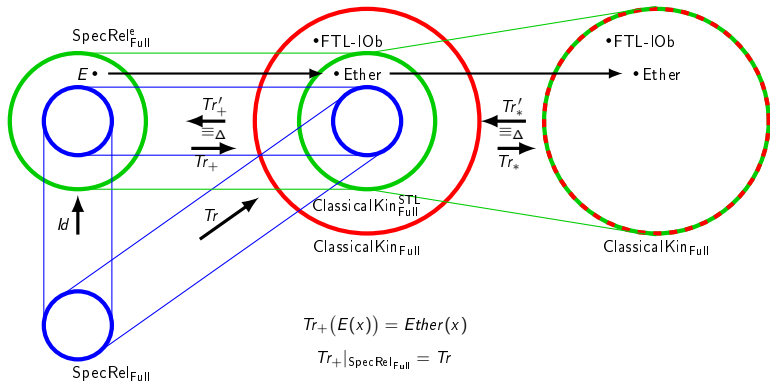
$Tr'_* \circ Tr_*(\text{ClassicalKin}_{Full}^{STL}) \Leftrightarrow \text{ClassicalKin}_{Full}^{STL}$ and

$Tr_* \circ Tr'_*(\text{ClassicalKin}_{Full}) \Leftrightarrow \text{ClassicalKin}_{Full}$



Theorem:

$Tr_* \circ Tr_+$ and $Tr'_+ \circ Tr'_*$ are a definitional equivalence between $SpecRel_{Full}^e$ and $ClassicalKin_{Full}$.



$Conceptual_distance(SpecRel_{Full}, ClassicalKin_{Full}) = 1$

Using Cylindric Algebras / Concept Algebras:

If $\mathfrak{A} \subset \mathfrak{B}$, then

$$d(\mathfrak{A}, \mathfrak{B}) = \min \{ n : \exists b_1 \dots b_n \text{ such that } \langle \mathfrak{A}, b_1 \dots b_n \rangle = \mathfrak{B} \}$$

where

- $\mathfrak{A} \subset \mathfrak{B}$ means that \mathfrak{A} is embedable in \mathfrak{B} ,
- $b_1 \dots b_n$ are the minimal elements to be added to \mathfrak{A} to generate \mathfrak{B} .

For arbitrary algebras \mathfrak{A} and \mathfrak{B} :

$$d(\mathfrak{A}, \mathfrak{B}) = \min \left\{ \sum_{i=0}^N d(\mathfrak{B}_i, \mathfrak{B}_{i+1}) : \mathfrak{B}_0 = \mathfrak{A}, \mathfrak{B}_i \sim \mathfrak{B}_{i+1}, \mathfrak{B}_N = \mathfrak{B} \right\},$$

where \sim means \subset or \supset .

To do:

- Establish the relation between the ad-hoc approach and the algebraic definition:
 - Algebraic definition: only adding primitive concepts,
 - Ad-hoc approach: also removing primitive or defined concepts.
- Prove that conceptual distance is indeed a metric.
- Can the conceptual distance be infinite?
 - The maximum distance between two theories: remove all concepts from the first theory and add all the the concepts for the second theory.
 - The distance can only be infinite if (at least) one of the theories has an infinite number of concepts.