Comparing Theories in First Order Logic

joint work with Gergely Székely
and the "Andréka–Németi School"

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Introduction

- Traditionally, *empirical adequacy* is the criterion to compare competing theories.
- However, this is not always straightforward:
  - Early Copernican heliocentrism provided less accurate predictions than the old Prolemaic geocentric model;
    - Reference: Thomas Kuhn, “The Copernican Revolution: Planetary Astronomy in the Development of Western Thought”, Harvard University Press 1957
  - One theory may be better suited to explain one part of the empirical data, another theory may be better suited to explain other parts of reality, e.g. relativity theory vs. quantum mechanics.
- Other approaches: e.g. historical – how did the theories evolve in time and influence each other, text analysis, etc...
- **Let us approach the comparison of theories qualitatively and quantitatively on formal grounds.**

Overview

- Epistemological context: the “Andréka–Németi School”
- Mathematical tools
- Kinds of equivalence, equivalent and non-equivalent theories

- Results on comparing classical and relativistic kinematics
- Conceptual distances between formal theories
- Work-in-progress: first steps in extending our results towards classical dynamics and general relativity theory: inelastic collisions
- Work-in-progress: variable independence of concepts

The "Andréka–Németi School or Tradition"

- Originated at the Algebraic Logic Group at the Alfréd Rényi Institute for Mathematics, Budapest
- Hajnal Andréka, István Németi, Judit X. Madarász, Gergely Székely, A. Andai, G. Sági, I. Sain, Cs. Tőke
  - "Big Book" a.k.a "Champagne":
    "On the logical structure of relativity theories"
    `https://old.renyi.hu/pub/algebraic-logic/Contents.html`
- Mike Stannett, Michèle Friend, Mohamed Khaled...
- Inspired by:
  - David Hilbert's sixth problem "Mathematical Treatment of the Axioms of Physics"
  - Logical empiricism of the Wiener Kreis
  - Alfred Tarski's initiative "Logic, Methodology and Philosophy of Science"

Mathematical tools

- Many-Sorted First-Order Logic
- Axiomatic method: theories are sets of axioms (and their consequences)
- Model Theory: the study of the relationship between formal theories (a collection of sentences in a formal language expressing statements about a mathematical structure), and their models (those structures in which the statements of the theory hold)
- Definability Theory: Translations, interpretations, and definitional equivalence
- Cylindric Algebra, Concept Algebra

Let $\mathfrak{M}$ be a model and φ be a formula of its language. Then the **meaning** *of* φ *in* $\mathfrak{M}$ is defined as the set of sequences from $\mathfrak{M}$ satisfying φ, i.e.,

$$[\![\varphi]\!]^{\mathfrak{M}} \stackrel{\text{def}}{=} \{\bar{a} \in M^{\omega} : \mathfrak{M} \models \varphi[\bar{a}]\}.$$

In general,

$$[\![\forall x \varphi]\!]^{\mathfrak{M}} \subseteq [\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\exists x \varphi]\!]^{\mathfrak{M}}.$$

Cylindric Algebra

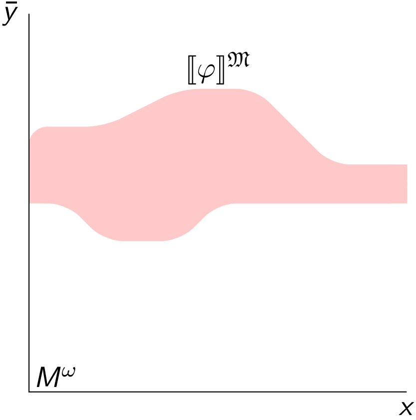


Figure: $[\![\varphi]\!]^{\mathfrak{M}} \subseteq M^{\omega}$.

Cylindric Algebra

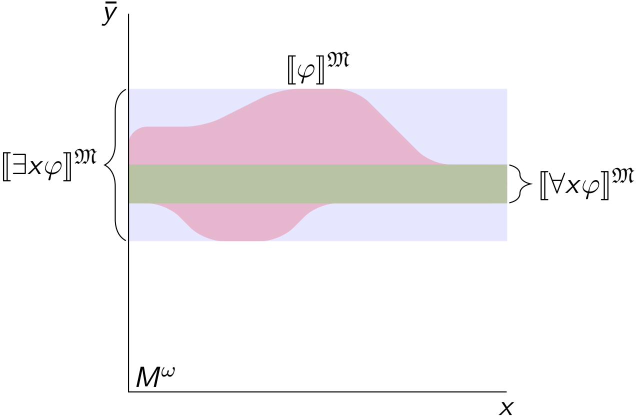


Figure: $[\![\forall x\varphi]\!]^{\mathfrak{M}} \subseteq [\![\varphi]\!]^{\mathfrak{M}} \subseteq [\![\exists x\varphi]\!]^{\mathfrak{M}}$.

Definability theory

Definition

A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

Definition

An *interpretation* of theory T_1 in theory T_2 is a translation Tr which translates all tautologies and all axioms of T_1 into theorems of T_2.

Definition

A *definitional equivalence* exists between two theories if those theories can be interpreted in each other and if all formulas from both theories translated into the other theory and back are *logical equivalent* to the original formulas.

Reference: Koen Lefever and Gergely Székely,
"On Generalization of Definitional Equivalence to Non-Disjoint Languages",
Journal of Philosophical Logic, Volume 48 Issue 4 pages 709-729, 30 August 2019.

Equivalent theories

- Logical equivalence
  - The consequences of the axioms of both theories are identical.
  - Only possible if both theories have the same language.
  - Example: geometry using the five axioms of Euclid, and geometry where Euclid's Fifth Axiom is replaced by Playfair's Axiom.
    - Reference: J. Playfair, "Elements of Geometry", W. E. Dean 1846.
- Definitional equivalence
  - Also possible if both theories have different languages.
  - Example: Boolean Algebras in the language $\wedge$, $\vee$, $\neg$, 0, 1 and the theory of Complemented Bounded Distributive Lattices in the language $\leq$.

Logical Equivalence – *too strict*

$\Downarrow \not\Uparrow$

Definitional Equivalence

$\Downarrow \not\Uparrow$

Conceptual Equivalence (Morita Equivalence, definitional equivalence with different sorts)

$\Downarrow \not\Uparrow$

Categorical Equivalence – *too lax*

Etcetera: bi-interpretability, mutual interpretability, sentential equivalence...

References:

- "*Morita Equivalence*" by Thomas William Barrett and Hans Halvorson in The Review of Symbolic Logic 9(3)
- "*Mutual definability does not imply definitional equivalence, a simple example*" by Andréka, H., Madarász, J. X., and Németi, I. in Mathematical Logic Quarterly, 2005.
- "*Categories of theories and interpretations*" by A. Visser, in 'Logic in Tehran', volume 26 of Lecture Notes in Logic, 2006
- "*Definable Categorical Equivalence*" by Laurenz Hudetz, 2018
- "*On Generalization of Definitional Equivalence to Non-Disjoint Languages* by K. Lefever and G. Székely, Journal of Philosophical Logic, Volume 48 Issue 4, 2019

Non-equivalent theories

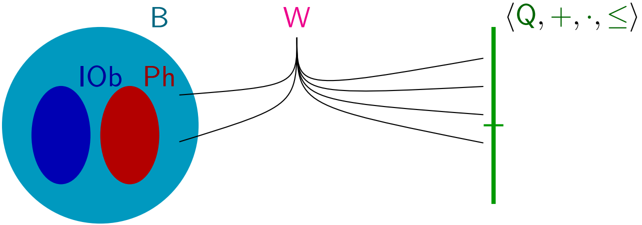
- It is usually easy to prove that two theories are not equivalent: one model which cannot be translated is enough.

- Qualitative: what do we need to add to / substract from a theory to make it equivalent to another theory?
- Quantitative: how far are two theories away from each other?

Previous results on classical and relativistic kinematics

- K. Lefever: "*Using Logical Interpretation and Definitional Equivalence to Compare Classical Kinematics and Special Relativity Theory*"
  PhD Dissertation, Vrije Universiteit Brussel (2017).
- K. Lefever and G. Székely: "*Comparing Classical and Relativistic Kinematics in First-Order Logic*", Logique et Analyse, ISSN: 2295-5836, p. 57-117, Vol 61, Nr 241 (2018).

Language: $\{\, \mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \mathsf{W} \,\}$

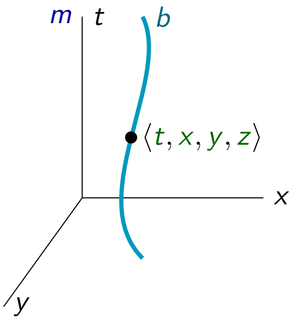


$\mathsf{B} \leftrightsquigarrow$ Bodies (things that move)
$\mathsf{IOb} \leftrightsquigarrow$ Inertial Observers $\quad$ $\mathsf{Ph} \leftrightsquigarrow$ Photons (light signals)

$\mathsf{Q} \leftrightsquigarrow$ Quantities
$+$, $\cdot$ and $\leq$ $\leftrightsquigarrow$ field operations and ordering

$\mathsf{W} \leftrightsquigarrow$ Worldview (a 6-ary relation of type BBQQQQ)

$\mathsf{W}(m, b, t, x, y, z) \leftrightsquigarrow$ " observer m coordinatizes body b at spacetime location $\langle t, x, y, z \rangle$."



Worldline of body b according to observer m

$$wl_m(b) := \{\langle t, x, y, z \rangle \in \mathrm{Q}^4 \;:\; \mathsf{W}(m, b, t, x, y, z)\}$$

Axioms for classical and relativistic kinematics

$$\mathsf{Kin} := \{\mathsf{AxEField}, \mathsf{AxEv}, \mathsf{AxSelf}, \mathsf{AxSymD}, \mathsf{AxLine}, \mathsf{AxTriv}, \mathsf{AxNoAcc}\}$$

$$\mathsf{ClassicalKin} := \mathsf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}_+\}$$

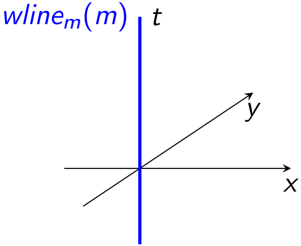
$$\mathsf{SpecRel} := \mathsf{Kin} \cup \{\mathsf{AxPh}_\mathsf{c}, \mathsf{AxThExp}\}$$

Kin:

- AxEField: The structure of quantities $\langle Q, +, \cdot, \leq \rangle$ is an Euclidean field.
- AxEv: Inertial observers coordinatize the same events (meetings of bodies).
- AxSelf: Every inertial observer is stationary according to himself.
- AxSymD: Inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them.
- AxLine: The worldlines of inertial observers are straight lines according to inertial observers.
- AxTriv: Any trivial transformation of an inertial observer is also an inertial observer.
- AxNoAcc: All observers are inertial observers.

AxSelf :

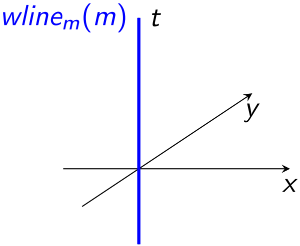
Every Inertial observer is stationary according to himself.



$$\forall mtxyz\ \Big(\mathsf{IOb}(m) \to \big[\mathsf{W}(m,m,t,x,y,z) \leftrightarrow x = y = z = 0\big]\Big).$$

AxSelf :

Every Inertial observer is stationary according to himself.



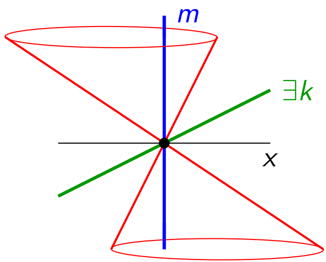
$$\forall mtxyz\ \Big(\mathsf{IOb}(m) \to \big[\mathsf{W}(m,m,t,x,y,z) \leftrightarrow x = y = z = 0\big]\Big).$$

$$\forall m\bar{x}\ \Big(\mathsf{IOb}(m) \to \big[\mathsf{W}(m,m,\bar{x}) \leftrightarrow x_1 = x_2 = x_3 = 0\big]\Big).$$

ClassicalKin:

- AxAbsTime: The time difference between two events is the same for all inertial observers.
- AxEther: There exists an inertial observer in which the light cones are right.
- $\mathsf{AxThExp}_+$: Inertial observers can move along any non-horizontal straight line:

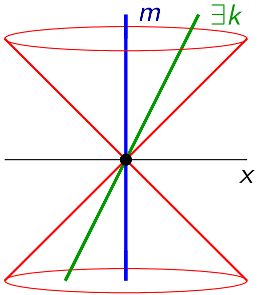
$$(\exists h \in B)\big[IOb(h)\big] \wedge$$
$$(\forall m \in IOb)\big(\forall \bar{x}, \bar{y} \in \mathrm{Q}^4\big)\ \big(x_0 \neq y_0 \rightarrow \big(\exists k \in IOb\big)\big[\bar{x}, \bar{y} \in wl_k(k)\big]\big)\ .$$



SpecRel:

- AxPh: For any inertial observer, the speed of light is the same in every direction everywhere, and it is finite. Furthermore, it is possible to send out a light signal in any direction.
- AxThExp: Inertial observers can move with any speed slower than that of light:

$$(\exists h \in B)\big[IOb(h)\big] \wedge (\forall m \in IOb)\big(\forall \bar{x}, \bar{y} \in Q^4\big)$$
$$\big(space(\bar{x}, \bar{y}) < \mathfrak{c} \cdot time(\bar{x}, \bar{y}) \rightarrow (\exists k \in IOb)\big[\bar{x}, \bar{y} \in wl_k(k)\big]\big).$$



Justification of the axioms

Kin:={AxEField, AxEv, AxSelf, AxSymD, AxLine, AxTriv, AxNoAcc}

ClassicalKin:=Kin $\cup$ {AxEther, AbsTime, $\mathsf{AxThExp}_{+}$}

SpecRel:= Kin $\cup$ {$\mathsf{AxPh}_{\mathsf{c}}$, AxThExp}

Theorem:

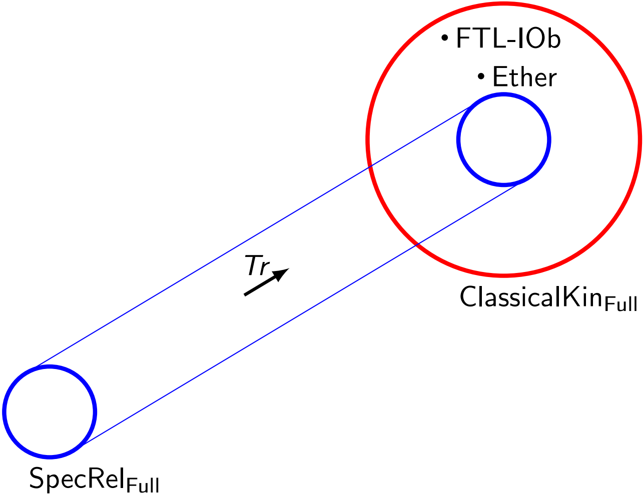
ClassicalKin $\vdash$ Worldview transformations are Galilean transformations.

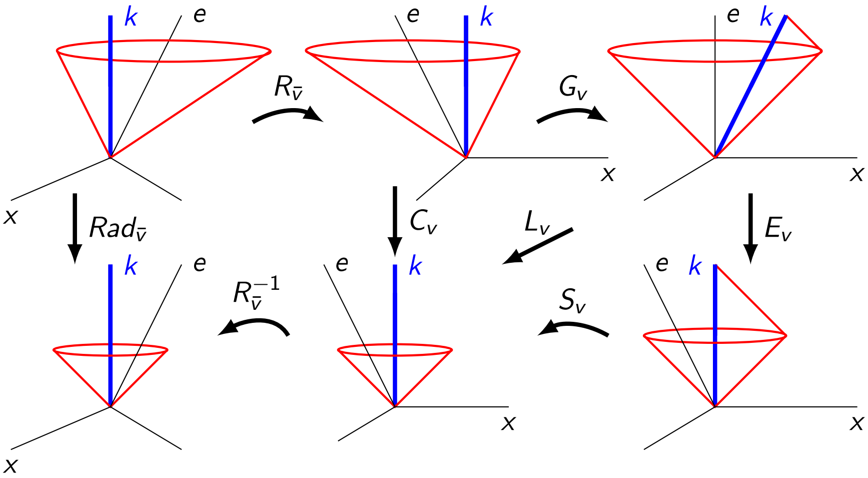
Theorem: (Andréka–Madarász–Németi, 1998)

SpecRel $\vdash$ Worldview transformations are Poincaré transformations.

Theorem:

There is an interpretation Tr of SpecRel_{Full} in $\text{ClassicalKin}_{Full}$.





$$Rad_{\bar{v}} = R_{\bar{v}}^{-1} \circ S_v \circ E_v \circ R_{\bar{v}} \circ G_{\bar{v}}$$

$$Tr(a+b=c) \stackrel{\text{def}}{\equiv} a+b=c$$

$$Tr(a \cdot b=c) \stackrel{\text{def}}{\equiv} a \cdot b=c$$

$$Tr(a<b) \stackrel{\text{def}}{\equiv} a<b$$

$$Tr\big(\mathsf{W}^{SR}(k,b,\bar{x})\big) \stackrel{\text{def}}{\equiv} \forall e[Ether(e) \rightarrow \mathsf{W}^{CK}\big(k,b,Rad^{-1}_{\bar{v}_k(e)}(\bar{x})\big)]$$

$$Tr\big(\mathsf{IOb}^{SR}(k)\big) \stackrel{\text{def}}{\equiv} \mathsf{IOb}^{CK}(k) \wedge \forall e[Ether(e) \rightarrow speed^{CK}_{e}(k) < \mathfrak{c}_{\mathfrak{e}}]$$

where

$$e \in Ether \stackrel{def}{\Longleftrightarrow} Ether(e) \stackrel{def}{\Longleftrightarrow} \mathsf{IOb}^{CK}(e) \wedge \forall p[Ph(p) \rightarrow speed^{CK}_{e}(p) = \mathfrak{c}_{\mathfrak{e}}]$$

Example: Translation of AxSelf^{SR}

AxSelf :

Every Inertial observer is stationary according to himself.

$$(\forall m \in IOb^{SR})(\forall \bar{x} \in Q^4)\left[W^{SR}(m, m, \bar{x}) \leftrightarrow x_1 = x_2 = x_3 = 0\right]$$

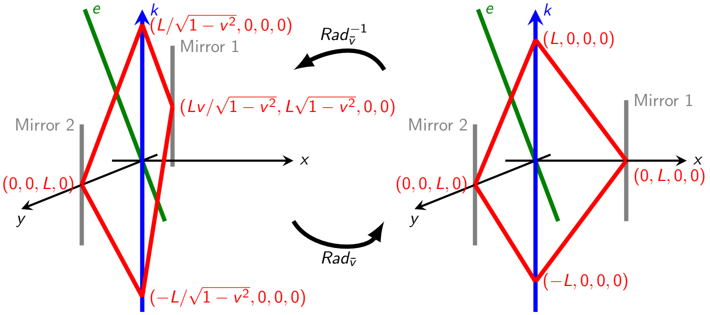
$$Tr(\text{AxSelf}^{SR}) \equiv$$

$$\forall m\Big(IOb^{CK}(m) \wedge (\forall e \in Ether)\big(speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big) \\ \rightarrow (\forall \bar{x} \in Q^4)\,(\forall e \in Ether)\left[W^{CK}(m, m, Rad^{-1}_{\bar{v}_k(e)}(\bar{x})) \leftrightarrow x_1 = x_2 = x_3 = 0\right]\Big)$$

$$\equiv$$

$$(\forall m \in IOb^{CK})(\forall e \in Ether)\Big(speed_e(m) < \mathfrak{c}_{\mathfrak{e}} \\ \rightarrow (\forall \bar{x} \in Q^4)\left[W^{CK}(m, m, Rad^{-1}_{\bar{v}_m(e)}(\bar{x})) \leftrightarrow x_1 = x_2 = x_3 = 0\right]\Big)$$

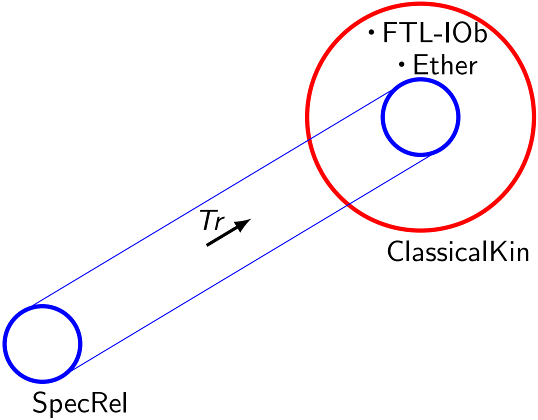
Example: the Michelson–Morley Experiment



Note that while our translation function *Tr* translates axioms of special relativity theory into theorems of classical kinematics, models are transformed the other way round from classical mechanics to special relativity theory.

Theorem:

There is no definitional equivalence between SpecRel and ClassicalKin.



How can we make SpecRel and ClassicalKin equivalent?

Adding and removing concepts to make both theories equivalent:

- Removing FTL observers from ClassicalKin
- Adding a "primitive ether" to SpecRel

$\mathsf{ClassicalKin}^{STL} :=$
$\mathsf{Kin} \cup \{\mathsf{AxEther}, \mathsf{AbsTime}, \mathsf{AxThExp}^{\mathsf{STL}}, \mathsf{AxNoFTL}\}$

AxNoFTL :

All inertial observers move slower than light with respect to the ether frames.

$$\neg\exists m\big(IOb(m) \land \exists e[Ether(e) \land Speed_e^{CK}(m) \geq \mathfrak{c}_{\mathfrak{e}}]\big).$$

$\mathsf{AxThExp}^{\mathsf{STL}}$:

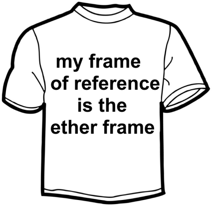
Inertial observers can move with any speed which is in the ether frame slower than that of light.

$$\exists h\ \big(IOb(h)\big) \land \forall e\bar{x}\bar{y}\big(Ether(e) \land \mathrm{space}(\bar{x},\bar{y}) < \mathfrak{c}_{\mathfrak{e}} \cdot \mathrm{time}(\bar{x},\bar{y}) \\ \rightarrow \exists k\ \ IOb(k) \land W(e,k,\bar{x}) \land W(e,k,\bar{y})\big).$$

$$\mathsf{SpecRel}^{e} := \mathsf{SpecRel} \cup \{\mathsf{AxPrimitiveEther}\}$$

AxPrimitiveEther :

There is a non-empty class of ether observers, stationary with respect to each other, which is closed under trivial transformations.

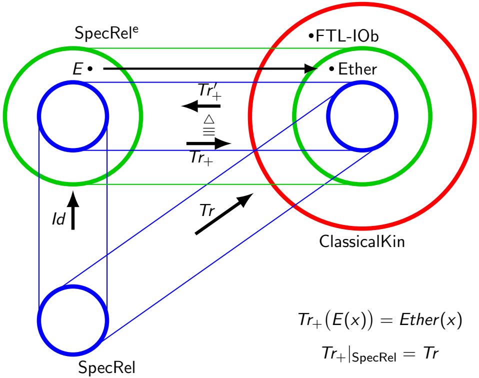


$$\exists e\Big(E(e) \land \forall k\Big[[IOb(k) \land (\exists T \in Triv) w^{SR}_{ek} = T] \leftrightarrow E(k)\Big]\Big)$$

Theorem:

There is an interpretation Tr'_+ of ClassicalKin^{STL} in SpecRel^e.

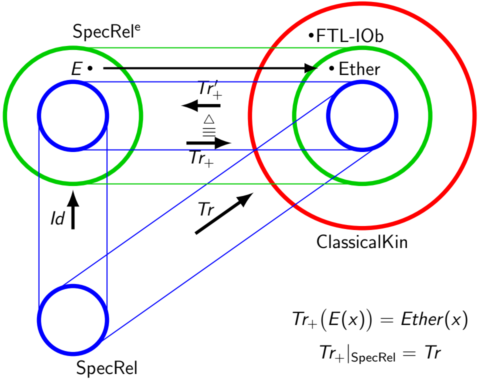
$$SpecRel^e \;\vdash\; Tr'_+(ClassicalKin^{STL})$$



Theorem:

Tr_+ and Tr'_+ are a definitional equivalence between $\mathsf{SpecRel}^e$ and $\mathsf{ClassicalKin}^{STL}$.

$Tr'_+ \circ Tr_+(\mathsf{SpecRel}^e) \Leftrightarrow \mathsf{SpecRel}^e$ and $Tr_+ \circ Tr'_+(\mathsf{ClassicalKin}^{STL}) \Leftrightarrow \mathsf{ClassicalKin}^{STL}$



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\mathsf{SpecRel}} = Tr$$

Theorem:

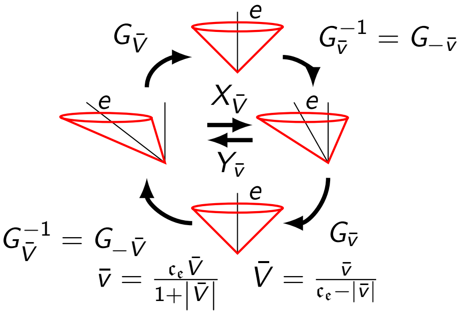
There is an interpretation Tr_* of ClassicalKin^{STL} in ClassicalKin.

$$\mathit{ClassicalKin} \vdash Tr_*(\mathit{ClassicalKin}^{STL})$$

Theorem:

There is an interpretation Tr'_* of ClassicalKin in ClassicalKin^{STL}.

$$\mathit{ClassicalKin}^{STL} \vdash Tr'_*(\mathit{ClassicalKin})$$

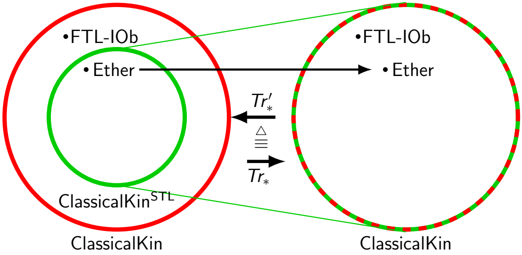


Theorem:

Tr_* and Tr'_* are a definitional equivalence between ClassicalKin^{STL} and ClassicalKin.

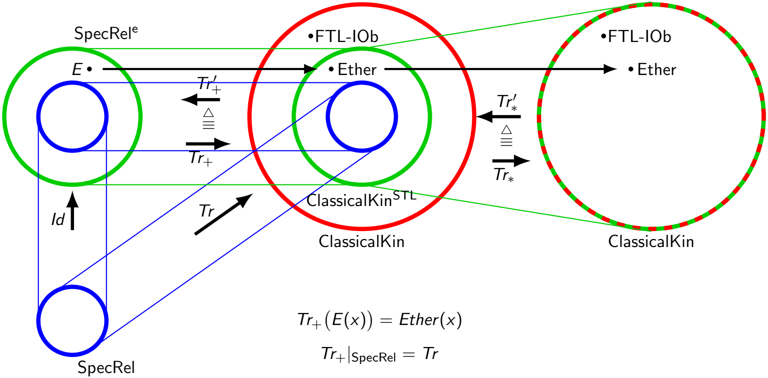
$Tr'_* \circ Tr_*(\text{ClassicalKin}^{STL}) \Leftrightarrow \text{ClassicalKin}^{STL}$ and

$Tr_* \circ Tr'_*(\text{ClassicalKin}) \Leftrightarrow \text{ClassicalKin}$



Theorem:

SpecRel^e and ClassicalKin are definitionally equivalent.



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\text{SpecRel}} = Tr$$

Conceptual Distance

- Mohamed Khaled, G. Székely, Koen Lefever, and Michèle Friend: "*Distances between formal theories*", The Review of Symbolic Logic, Volume 14 Issue 3 pages 633-654, September 2020.

Equivalence relations gives us a trivial (discrete) distance:

Let X-be any set of theories and E any equivalence relation on X. The *discrete distance* on (X, E) is the following:

$$d(x,y) = \begin{cases} 0 & \text{if } E(x,y), \\ 1 & \text{if } \neg E(x,y). \end{cases}$$

We define *distances* between two theories based on the minimum number of things (e.g., axioms or concepts) which need to be added to or substracted from one theory to make it (logically, definitionally, etc.) equivalent to the other theory.

Two theories which are equivalent have a distance of zero.

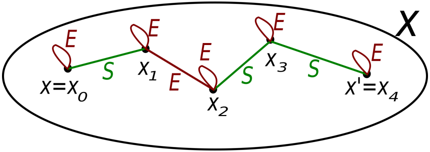
Definition

A **cluster network** is a triple (X, E, S), where X is a class with an equivalence relation E and a symmetric relation S.

$E(x, x')$: x and x' are basically the same objects.

$S(x, x')$: x or x' can be reached from the other in one "step."

A **path** *between* x *and* x' *in cluster network* (X, E, S) is a finite sequence of joining E-edges and S-edges connecting x and x'.



The **length** *of a path* is the number of S-edges in the path.

Objects $x, x' \in X$ *are* **connected** *in* (X, E, S) iff there is a path from one of them to the other.

Let $\mathcal{X} = (X, E, S)$ be a cluster network.

Definition

The **step distance** *on* $\mathcal{X}$ is the function $d_{\mathcal{X}} : X \times X \to \mathbb{N} \cup \{\infty\}$ defined as:

$$d_{\mathcal{X}}(x, x') \stackrel{\text{def}}{\equiv} \min\{k \in \mathbb{N} : \exists \text{ a path from } x \text{ to } x' \text{ of length } k\}$$

if x and x' are connected in $\mathcal{X}$, and

$$d_{\mathcal{X}}(x, x') \stackrel{\text{def}}{\equiv} \infty \quad \text{otherwise}$$

for each $x, x' \in X$.

Theorem:

Let $d_{\mathcal{X}} : X \times X \to \mathbb{N} \cup \{\infty\}$ be the step distance on cluster network $\mathcal{X} = (X, E, S)$. Then for each $x, y, z \in X$,

(a) $d_{\mathcal{X}}(x, y) \geq 0$, and $d_{\mathcal{X}}(x, y) = 0 \iff x \, E \, y$.

(b) $d_{\mathcal{X}}(x, y) = d_{\mathcal{X}}(y, x)$.

(c) $d_{\mathcal{X}}(x, y) \leq d_{\mathcal{X}}(x, z) + d_{\mathcal{X}}(z, y)$.

Definition

Theory T' of language $\mathcal{L}'$ is a **one-concept-extension** *of theory T of language $\mathcal{L}$:*

$$T \rightsquigarrow T' \overset{def}{\iff} \mathcal{L}' = \mathcal{L} \cup \{R\} \text{ and } (\forall \varphi \in \mathcal{L})(T' \models \varphi \text{ iff } T \models \varphi)$$

for some relation symbol R.

$$T \leftrightsquigarrow T' \overset{def}{\iff} T \rightsquigarrow T' \text{ or } T' \rightsquigarrow T$$

Definition

Let $\mathcal{T}$ be a class of theories. The **conceptual distance** $\mathrm{Cd}_{\mathcal{T}}$ is the step distance on cluster network $(\mathcal{T}, \stackrel{\triangle}{\equiv}, \leftrightsquigarrow)$, where $\stackrel{\triangle}{\equiv}$ is the definitional equivalence.

If $\mathcal{T}$ is the class of all theories of FOL, we omit the subscript $\mathcal{T}$ and simply write Cd.

Definition

Let $\mathcal{T}$ be a class of theories. The **conceptual distance** $\mathrm{Cd}_{\mathcal{T}}$ is the step distance on cluster network $(\mathcal{T}, \stackrel{\triangle}{\equiv}, \leftrightsquigarrow)$, where $\stackrel{\triangle}{\equiv}$ is the definitional equivalence.

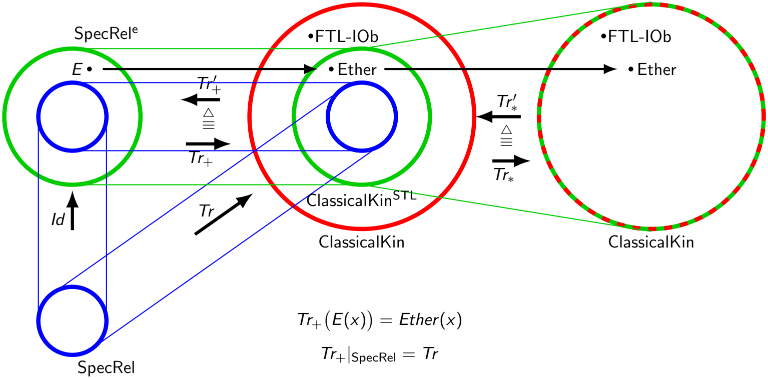
If $\mathcal{T}$ is the class of all theories of FOL, we omit the subscript $\mathcal{T}$ and simply write Cd.

Theorem:

For every $n \in \mathbb{N} \cup \{\infty\}$, there are theories T and T' such that $\mathrm{Cd}(T, T') = n$.

Theorem:

$$\mathrm{Cd}(\mathrm{ClassicalKin}, \mathrm{SpecRel}) = 1$$



$$Tr_+(E(x)) = Ether(x)$$

$$Tr_+|_{\mathrm{SpecRel}} = Tr$$

Proof.:

$$SpecRel \rightsquigarrow SpecRel^e \triangleq ClassicalKin^{STL} \triangleq ClassicalKin$$

Q.E.D.

Extending our results to dynamics

Work in progress

Inelastic collisions provide a simple case study, we need the following new concepts on top of those from kinematics:

- Mass: this also gives us impuls $\bar{P} = m\bar{v}$ and force $\bar{F} = m\bar{a}$
- Inertial bodies and inertial particles
- incoming and outgoing bodies at inelastic collisions
- New Axioms:
  - AxMass
  - AxSpeed
  - AxColl

Language: $\{\, \mathsf{B}, \mathsf{IOb}, \mathsf{Ph}, \mathsf{Q}, +, \cdot, \leq, \mathsf{W}, \mathsf{M} \,\}$

Definition

M (the mass relation) is a 3-place relation symbol the first two arguments of which are of sort B and the third argument is of sort Q, reading $\mathsf{M}(k, b, q)$ as "the mass of body b is q according to observer k."

Definition

The **relativistic mass** of body b according to inertial observer k, in symbols $m_k(b)$, is defined to be q if $\mathsf{M}(k, b, q)$ holds and there is only one such $q \in \mathsf{Q}$; otherwise $m_k(b)$ is undefined.

Definition

If all inertial observers k which are stationary with respect to body b agree on its mass, then $m_0(b) = m_k(b)$ is the rest mass of body b.

From this we can define 4-velocity V and 4-momentum P:

Definition

$P_k(b) = \big(\mathfrak{c}_{\mathfrak{e}} m_k(b), m_k(b)\bar{v}_k(b)\big) = m_0(b)V_k(b)$.

Body b is an **inertial body** if for all inertial obsevers k, the worldline $wl_k(b)$ is a subset of a straight line:

Definition

$$\mathsf{IB}(b) \stackrel{def}{\iff} (\forall k \in \mathsf{IOb})\left(\exists \bar{x}, \bar{y} \in Q^4\right)\left(\forall \bar{z} \in Q^4\right)\Big(W(k, b, \bar{z})$$

$$\rightarrow (\exists a \in Q)(\bar{z} = a\bar{x} + (1 - a)\bar{y})\Big).$$

Body b is called **inertial particle** according to observer k, in symbols $IP_k(b)$, iff b is an inertial body and $m_k(b)$ is defined for inertial observer k:

Definition

$$IP_k(b) \stackrel{def}{\iff} \mathsf{IB}(b) \wedge \mathsf{IOb}(k) \wedge (\exists m \in Q)\Big(\mathsf{M}(k, b, m)$$

$$\wedge(\forall m' \in Q)\big(\mathsf{M}(k, b, m') \rightarrow m' = m\big)\Big).$$

Body b is **incoming** at coordinate point $\bar{x}$ according to inertial observer k, iff b is an inertial particle, $\bar{x}$ is on the world-line of b, and the time component of each coordinate point on the world-line of b different from $\bar{x}$ is *less* than the time component of $\bar{x}$:

Definition

$$in_k(b, \bar{x}) \overset{def}{\Longleftrightarrow}$$

$$IP_k(b) \land \mathsf{W}(k, b, \bar{x}) \land \forall \bar{y} \big(\mathsf{W}(k, b, \bar{y}) \rightarrow [\bar{y} = \bar{x} \lor y_1 < x_1]\big)$$

The definition for **outgoing** is similar:

Definition

$$out_k(b, \bar{x}) \overset{def}{\Longleftrightarrow}$$

$$IP_k(b) \land \mathsf{W}(k, b, \bar{x}) \land \forall \bar{y} \big(\mathsf{W}(k, b, \bar{y}) \rightarrow [\bar{y} = \bar{x} \lor y_1 > x_1]\big)$$

The **collision** of inertial particles $a_1 \dots a_n$ at some point $\bar{x}$ according to observer k, creating inertial particles $b_1 \dots b_m$ is:

Definition

$$coll_k(a_1 \dots a_n : b_1 \dots b_m)_{@\bar{x}} \overset{def}{\iff} \bigwedge_{i=1}^{n} in_k(a_i, \bar{x}) \wedge \bigwedge_{i=1}^{m} out_k(b_i, \bar{x}) \wedge$$

$$\sum_{i=1}^{n} m_k(a_i) = \sum_{i=1}^{m} m_k(b_i) \wedge \sum_{i=1}^{n} m_k(a_i) \cdot v_k(a_i) = \sum_{i=1}^{m} m_k(b_i) \cdot v_k(b_i).$$

When we do not care where the collision occurs, we can drop the $@\bar{x}$ subscript:

Definition

$$coll_k(a_1 \dots a_n : b_1 \dots b_n) \overset{def}{\iff}$$

$$(\exists \bar{x} \in Q^4)\ coll_k(a_1 \dots a_n : b_1 \dots b_m)_{@\bar{x}}$$

AxMass :

If the relativistic masses and velocities of two inertial particles coincide for an inertial observer, then their relativistic masses coincide for every inertial observer

$$IOb(k) \land IOb(h) \land Ip(a) \land Ip(b) \land m_k(a) = m_k(b) \land v_k(a) = v_k(b) \\ \rightarrow m_h(a) = m_h(b);$$

AxSpeed :

If an inertial particle is moving with the same slower than light speed according to two inertial observers, then the relativistic masses of the particle are the same for them

$$IOb(k) \land IOb(h) \land Ip(b) \land v_k(b) < \mathfrak{c} \rightarrow m_k(b) = m_h(b);$$

AxColl :

For every coordinate point $\bar{x} \in Q^n$ and for all positive quantities $m_1 \ldots m_n, m'_1 \ldots m'_l$ and 3-vectors $\bar{v}_1, \ldots, \bar{v}_n, \bar{v}'_1 \ldots \bar{v}'_l \in Q^3$ such that

$$\sum_{i=1}^{n} m_i = \sum_{j=1}^{l} m'_j \quad \text{and} \quad \sum_{i=1}^{n} m_i \cdot \bar{v}_i = \sum_{j=1}^{l} m'_j \cdot \bar{v}'_j,$$

there are inertial particle $b_1 \ldots b_n$ and $b'_1 \ldots b'_l$ such that $m_k(b_i) = m_i$, $\bar{v}_k(b_i) = \bar{v}_i$, $m_k(b'_j) = m'_j$, and $v_k(b'_j) = v'_j$ for all $i \leq n$ and $j \leq l$, and $coll_k(b_1 \ldots b_n : b'_1 \ldots b'_l)@\bar{x}$.

Let us define $m_k^*(b)$ from $P_k^*(b)$ as $m_k^*(b) \stackrel{def}{=} \frac{P_k^*(b)_0}{\mathfrak{c}_\mathfrak{e}}$ and define $P_k^*(b)$ as the image of $P_k(b)$ by $Rad_{v_k(e)}$.
Then by linearity of Rad_v, we have $P_k^*(a) + P_k^*(b) = P_k^*(c)$ if $P_k(a) + P_k(b) = P_k(c)$.
This allows us to tranlate mass as follows:

Definition

$$Tr\big(\mathsf{M}^{sr}(k, b, m)\big) \stackrel{def}{=}$$
$$(\forall e \in Ether)(\exists m' \in Q)$$
$$\left(\mathsf{M}^{ck}(k, b, m') \wedge m = \frac{Rad_{\bar{v}_k^{ck}(e)}\big(\mathfrak{c}_\mathfrak{e} m', m' \bar{v}_k^{ck}(b)\big)_0}{\mathfrak{c}_\mathfrak{e}}\right).$$

Variable-independent concepts

Work in progress

Assuming ClassicalKin, all ether observers are stationary with respect to each other, and hence they agree on the speed of light.

Let $b, k_1, \ldots, k_n$ be variables of sort B . We say that formula φ is *ether-observer-independent* in variable b provided that $k_1, \ldots, k_n$ are inertial observers if the truth or falsehood of φ does not depend on to which ether we evaluated b if $k_1, \ldots, k_n$ are evaluated to inertial observers, that is:

Definition

$$EOI_b^{k_1,\ldots,k_n}[\varphi] \overset{def}{\iff}$$

$$ClassicalKin \models (\forall k_1, \ldots, k_n \in IOb)(\forall e, e' \in Ether)$$

$$[\varphi(e/b) \iff \varphi(e'/b)]$$

where $\varphi(e/b)$ means that b gets replaced by e in all free occurences of b in formula φ.

The following rules can be used to show the ether independence of complex formulas:

1. From $EOI_b^{k_1,\dots,k_n}[\varphi]$ follows $EOI_b^{k_1,\dots,k_n}[\neg\varphi]$.
2. If $*$ is a logical connective, then from $EOI_b^{k_1,\dots,k_n}[\varphi]$ and $EOI_b^{h_1,\dots,h_m}[\psi]$ follows $EOI_b^{k_1,\dots,k_n,h_1,\dots,h_m}[\varphi * \psi]$.
3. From $EOI_b^{k_1,\dots,k_n}[\varphi]$ follows $EOI_b^{k_1,\dots,k_n}[(\exists x \in Q)(\varphi)]$ and $EOI_b^{k_1,\dots,k_n}[(\exists h \in B)(\varphi)]$.
4. From $EOI_b^{k_1,\dots,k_n}[\varphi]$ follows $EOI_b^{k_1,\dots,k_n}[(\forall x \in Q)(\varphi)]$ and $EOI_b^{k_1,\dots,k_n}[(\forall h \in B)(\varphi)]$.

AxSelf :

Every Inertial observer is stationary according to himself.

$$(\forall k \in IOb^{SR})(\forall \bar{x} \in Q^4)\left[W^{SR}(k,k,\bar{x}) \leftrightarrow x_1 = x_2 = x_3 = 0\right]$$

$$Tr(\mathrm{AxSelf}^{\mathrm{SR}}) \equiv$$

$$\forall k \Big(IOb^{CK}(k) \wedge (\forall e \in Ether)\big(speed_e(k) < \mathfrak{c}_{\mathfrak{e}}\big)$$
$$\rightarrow (\forall \bar{x} \in Q^4)\,(\forall e \in Ether)\left[W^{CK}(k,k,Rad^{-1}_{\bar{v}_k(e)}(\bar{x})) \leftrightarrow x_1 = x_2 = x_3 = 0\right]\Big)$$

$$\equiv$$

$$(\forall k \in IOb^{CK})(\forall e \in Ether)\Big(speed_e(k) < \mathfrak{c}_{\mathfrak{e}}$$
$$\rightarrow (\forall \bar{x} \in Q^4)\left[W^{CK}(k,k,Rad^{-1}_{\bar{v}_k(e)}(\bar{x})) \leftrightarrow x_1 = x_2 = x_3 = 0\right]\Big)$$

Let $\mathfrak{M}$ be a model, and let φ and θ be formulas in the language of $\mathfrak{M}$. We say that φ is **independent of variable** v_i *in* $\mathfrak{M}$ iff for all sequence of elements $\bar{a} \in M^\omega$ and and $b \in M$,

$$\mathfrak{M} \models \varphi[\bar{a}] \iff \mathfrak{M} \models \varphi[\bar{a}^i_b],$$

where $\bar{a}^i_b$ denotes the sequence which is the same as $\bar{a}$ except at i where it is b, i.e., $\bar{a}^i_b = (a_0, \ldots, a_{i-1}, b, a_{i+1}, \ldots)$.

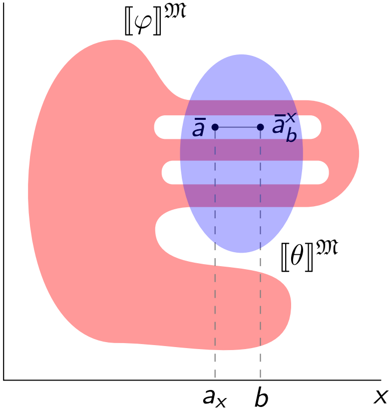
Proposition:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ be a formula of the language of $\mathfrak{M}$. Then the following statements are equivalent:

$$\begin{aligned}&\varphi \text{ is independent of } x \text{ in } \mathfrak{M}\\ &\qquad\iff [\![\forall x\varphi]\!]^{\mathfrak{M}} = [\![\varphi]\!]^{\mathfrak{M}}\\ &\qquad\qquad\iff [\![\varphi]\!]^{\mathfrak{M}} = [\![\exists x\varphi]\!]^{\mathfrak{M}}.\end{aligned}$$

We say that φ is **independent of variable** v_i *in* $\mathfrak{M}$ **provided** θ iff, for all sequence of elements $\bar{a} \in M^\omega$ and $b \in M$,

$$\mathfrak{M} \models \theta[\bar{a}] \text{ and } \mathfrak{M} \models \theta[\bar{a}^i_b] \implies (\, \mathfrak{M} \models \varphi[\bar{a}] \iff \mathfrak{M} \models \varphi[\bar{a}^i_b] \,)$$



Proposition:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ and θ be formulas of the language of $\mathfrak{M}$. Then the following statements are equivalent:

1. φ is independent of x in $\mathfrak{M}$ provided θ,
2. $[\![\theta \wedge \varphi]\!]^{\mathfrak{M}} = [\![\theta \wedge (\exists x \in \theta)\varphi]\!]^{\mathfrak{M}}$, and
3. $[\![\theta \rightarrow (\forall x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![\theta \rightarrow \varphi]\!]^{\mathfrak{M}}$.

Corollary:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ and ψ be formulas of the language of $\mathfrak{M}$. If ψ is independent of x in $\mathfrak{M}$, then

$$[\![\forall x(\varphi * \forall x \psi)]\!]^{\mathfrak{M}} = [\![\forall x(\varphi * \psi)]\!]^{\mathfrak{M}},$$

or in other words

$$\mathfrak{M} \models \forall x(\varphi * \forall x \psi) \leftrightarrow \forall x(\varphi * \psi).$$

Proposition:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ and θ be formulas of the language of $\mathfrak{M}$. Then the following statements are equivalent:

1. φ is independent of x in $\mathfrak{M}$ provided θ,
2. $[\![\theta \wedge \varphi]\!]^{\mathfrak{M}} = [\![\theta \wedge (\exists x \in \theta)\varphi]\!]^{\mathfrak{M}}$, and
3. $[\![\theta \to (\forall x \in \theta)\varphi]\!]^{\mathfrak{M}} = [\![\theta \to \varphi]\!]^{\mathfrak{M}}$.

Corollary:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ, ψ and θ be formulas of the language of $\mathfrak{M}$. If ψ is independent of x in $\mathfrak{M}$ provided θ, then

$$[\![\theta \wedge \varphi \to (\forall x \in \theta)\psi]\!]^{\mathfrak{M}} = [\![\theta \wedge \varphi \to \psi]\!]^{\mathfrak{M}}.$$

Corollary:

Let x a variable, let $\mathfrak{M}$ be a model, and let φ, ψ and $\iota(\bar{y})$, $\epsilon(x)$ be formulas of the language of $\mathfrak{M}$. If ψ is independent of x in $\mathfrak{M}$ provided $\iota \wedge \epsilon$, then

$$[\![\iota \wedge \epsilon \wedge \varphi \rightarrow (\forall x \in \epsilon)\psi]\!]^{\mathfrak{M}} = [\![\iota \wedge \epsilon \wedge \varphi \rightarrow \psi]\!]^{\mathfrak{M}},$$

and hence

$$\mathfrak{M} \models (\forall \bar{y} \in \iota)(\forall x \in \epsilon)(\varphi \rightarrow (\forall x \in \epsilon)\psi)$$

$$\leftrightarrow (\forall \bar{y} \in \iota)(\forall x \in \epsilon)(\varphi \rightarrow \psi).$$

This is the generalized version of what we need to simplify our translated formulas: ι can represent the set of inertial observers, and ϵ can represent the set of ether frames.