

Comparing Classical and Relativistic Kinematics in First-Order Logic

and introducing work-in-progress:
Conceptual Distances between Theories

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A *translation* is a function between formulas of languages preserving the logical connectives, i.e. $Tr(\phi \wedge \psi) = Tr(\phi) \wedge Tr(\psi)$, etc.

An *interpretation* of theory **Th₁** in theory **Th₂** is a translation Tr which translates all tautologies and all axioms of **Th₁** into theorems of **Th₂**.

A *definitional equivalence* exists between two theories if those theories can be interpreted in each other and if all formulas from both theories translated into the other theory and back are logical equivalent to the original formulas.

We use these concepts to show the *differences*
between theories which are *not* equivalent.

There are translations Tr , Tr_+ , and Tr'_+ between the languages of Classical Kinematics and Special Relativity Theory such that:

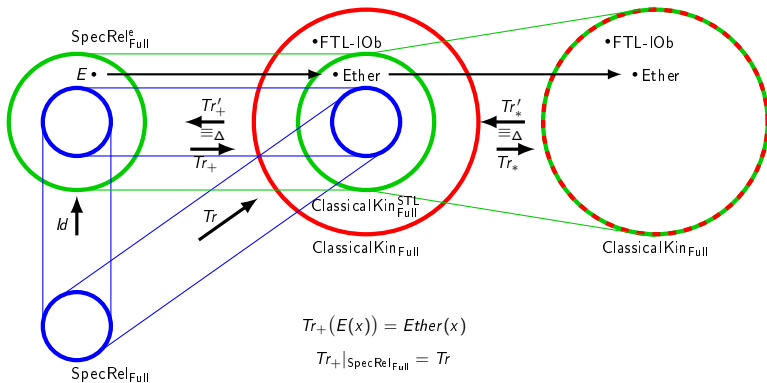
- $ClassicalKin_{Full} \vdash Tr(SpecRel_{Full})$
- $ClassicalKin_{Full}^{STL} \vdash Tr_+(SpecRel_{Full}^e)$
- $SpecRel_{Full}^e \vdash Tr'_+(ClassicalKin_{Full}^{STL})$
- Definitional equivalence: $SpecRel_{Full}^e \equiv_{\Delta} ClassicalKin_{Full}^{STL}$,
i.e., Tr_+ and Tr'_+ are inverses of each other up to logical
equivalence in $SpecRel_{Full}^e$ and $ClassicalKin_{Full}^{STL}$.

There are translations Tr_* , and Tr'_* between the languages of Classical Kinematics and Classical Kinematics restricted to Slower-Than-Light observers such that:

- $ClassicalKin_{Full}^{STL} \vdash Tr_*(ClassicalKin_{Full})$
- $ClassicalKin_{Full} \vdash Tr'_*(ClassicalKin_{Full}^{STL})$
- Definitional equivalence: $ClassicalKin_{Full} \equiv_{\Delta} ClassicalKin_{Full}^{STL}$,

and hence by transitivity of definitional equivalence:

- $SpecRel_{Full}^e \equiv_{\Delta} ClassicalKin_{Full}$.



We define the *Conceptual Distance* between two theories as the minimum number of concepts (i.e. logical predicates) which need to be added to or subtracted from one theory to make it definitionally (or logically, categorically, etc. . .) equivalent to the other theory.

Two theories which are equivalent have a Conceptual Distance of zero.

Adding a concept:

- introduce new primitive predicate R
- add axiom $\exists \bar{x}(\varphi_R(\bar{x}))$

Removing concept $\varphi(\bar{x})$:

- add axiom $\neg \exists \bar{x}(\varphi(\bar{x}))$
- change theory T to a weaker theory T^- to avoid contradictions

$$\mathbf{Kin} := \{\text{AxEField}, \text{AxEv}, \text{AxSelf}, \text{AxSymD}, \text{AxLine}, \text{AxTriv}, \text{AxNoAcc}\}$$
$$\mathbf{ClassicalKin}_{Full} := \mathbf{Kin} \cup \{\text{AxEther}, \text{AbsTime}, \text{AxThExp}_+\}$$
$$\mathbf{SpecRel}_{Full} := \mathbf{Kin} \cup \{\text{AxPh}_c, \text{AxThExp}\}$$

AxEField :

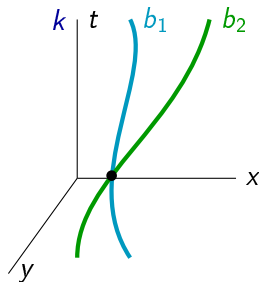
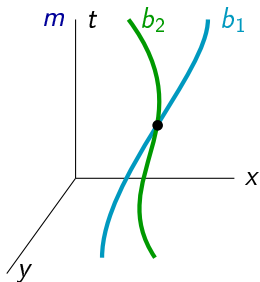
The **structure of quantities** $\langle \mathbb{Q}, +, \cdot, \leq \rangle$ is an *Euclidean field*,

- Real numbers: \mathbb{R} ,
- Real algebraic numbers: $\overline{\mathbb{Q}} \cap \mathbb{R}$,
- Hyperreal numbers: \mathbb{R}^* ,
- Real constructable numbers,
- Etc...

AxEv :

Inertial observers coordinatize the same events (meetings of bodies).

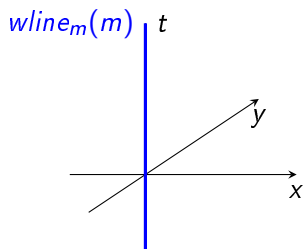
$$ev_m(\bar{x}) := \{b : W(m, b, \bar{x})\}$$



$$\forall m k \bar{x} [IOb(m) \wedge IOb(k) \rightarrow \exists \bar{y} ev_m(\bar{x}) = ev_k(\bar{y})].$$

AxSelf :

Every *Inertial observer* is stationary according to *himself*.

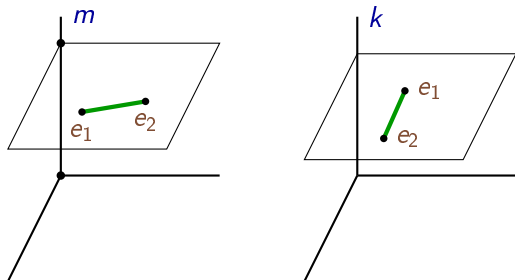


$$\forall mtxyz \left(IOb(m) \rightarrow [W(m, m, t, x, y, z) \leftrightarrow x = y = z = 0] \right).$$

AxSymD :

Inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them.

$$\text{space}(\bar{x}, \bar{y}) := \sqrt{(x_2 - y_2)^2 + \dots + (x_d - y_d)^2}$$



$$\forall mk \bar{x} \bar{y} \bar{x}' \bar{y}' [IOb(m) \wedge IOb(k) \wedge x_1 = y_1 \wedge x'_1 = y'_1 \wedge ev_m(\bar{x}) = ev_k(\bar{x}') \wedge ev_m(\bar{y}) = ev_k(\bar{y}') \rightarrow \text{space}(\bar{x}, \bar{y}) = \text{space}(\bar{x}', \bar{y}')]]$$

AxLine :

The worldlines of inertial observers are straight lines according to inertial observers.

$$\forall mk\bar{x}\bar{y}\bar{z} \left(IOb(m) \wedge IOb(k) \wedge W(m, k, \bar{x}) \wedge W(m, k, \bar{y}) \wedge W(m, k, \bar{z}) \right. \\ \left. \rightarrow \exists a [\bar{z} - \bar{x} = a(\bar{y} - \bar{x}) \vee \bar{y} - \bar{z} = a(\bar{z} - \bar{x})] \right).$$

(In *SpecRel*, AxLine is a theorem, so including it as an axiom is redundant.)

AxTriv :

Any trivial transformation of an inertial observer is also an inertial observer.

$\forall T \in Triv[\forall m \exists k(w_{mk} = T)]$, where Triv is the set of Trivial transformations, i.e. transformations that are isometries on space and translations on time.

Let us define an *observer* as any body which can coordinatize other bodies:

$$Ob(k) \stackrel{def}{\iff} \exists b \in B, \exists \bar{x} \in Q^4(W(k, b, \bar{x})).$$

AxNoAcc :

All observers are inertial observers.

$$\forall k \in B [W(Ob(k)) \rightarrow IOb(k)].$$

AxAbsTime :

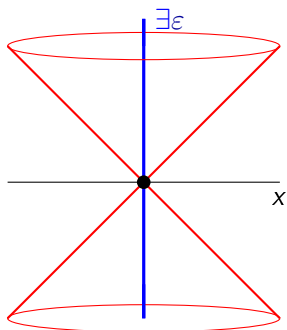
The time difference between two *events* is the same for all *inertial observers*.

$$\text{time}(\bar{x}, \bar{y}) := |x_1 - y_1|$$

$$\forall mk\bar{x}\bar{y}\bar{x}'\bar{y}' [IOb(m) \wedge IOb(k) \wedge ev_m(\bar{x}) = ev_k(\bar{x}') \wedge ev_m(\bar{y}) = ev_k(\bar{y}') \\ \rightarrow \text{time}(\bar{x}, \bar{y}) = \text{time}(\bar{x}', \bar{y}')].$$

AxEther(Einstein's AxLight) :

There exists an *inertial observer* in which the *light cones* are *right*.

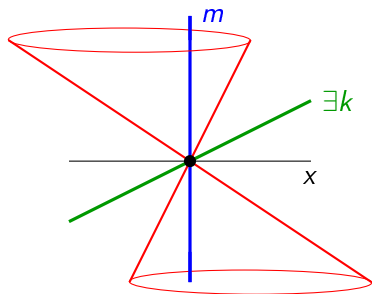


$$\exists \epsilon c \left[\text{IOb}(\epsilon) \wedge c > 0 \wedge \forall \bar{x} \bar{y} \left(\exists p \left[\text{Ph}(p) \wedge \text{W}(\epsilon, p, \bar{x}) \right. \right. \right. \\ \left. \left. \left. \wedge \text{W}(\epsilon, p, \bar{y}) \right] \leftrightarrow \text{space}(\bar{x}, \bar{y}) = c \cdot \text{time}(\bar{x}, \bar{y}) \right) \right]$$

AxThExp₊ :

Inertial observers can move along any non-horizontal straight line.

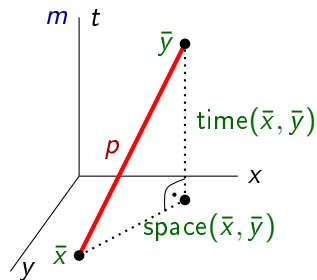
$$k \uparrow m \stackrel{\text{def}}{\iff} ev_k(\bar{x}) = ev_m(1, 0, 0, 0) \wedge ev_k(\bar{y}) = ev_m(0, 0, 0, 0) \rightarrow x_1 > y_1$$



$$\begin{aligned} \exists h (IOb(h)) \wedge \forall m \bar{x} \bar{y} (IOb(m) \wedge x_1 \neq y_1 \\ \rightarrow \exists k IOb(k) \wedge W(m, k, \bar{x}) \wedge W(m, k, \bar{y}) \wedge m \uparrow k). \end{aligned}$$

$AxPh_c$:

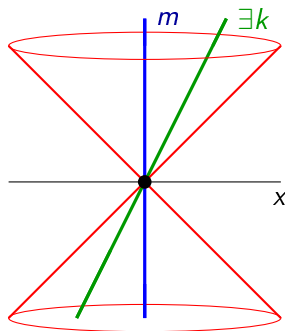
For any *inertial observer*, the *speed of light* is the same in every *direction everywhere*, and it is finite. Furthermore, it is possible to send out a *light signal* in any *direction*.



$$\exists c \left[c > 0 \wedge \forall m \bar{x} \bar{y} \left(\text{IOb}(m) \rightarrow \exists p \left[\text{Ph}(p) \wedge \text{W}(m, p, \bar{x}) \right. \right. \right. \\ \left. \left. \left. \wedge \text{W}(m, p, \bar{y}) \right] \leftrightarrow \text{space}(\bar{x}, \bar{y}) = c \cdot \text{time}(\bar{x}, \bar{y}) \right) \right]$$

AxThExp :

Inertial observers can move with any speed slower than that of light.



$$\begin{aligned} \exists h [IOb(h)] \wedge \forall m \bar{x} \bar{y} (IOb(m) \wedge \text{space}(\bar{x}, \bar{y}) < c \cdot \text{time}(\bar{x}, \bar{y}) \\ \rightarrow \exists k [IOb(k) \wedge W(m, k, \bar{x}) \wedge W(m, k, \bar{y}) \wedge m \uparrow k]). \end{aligned}$$

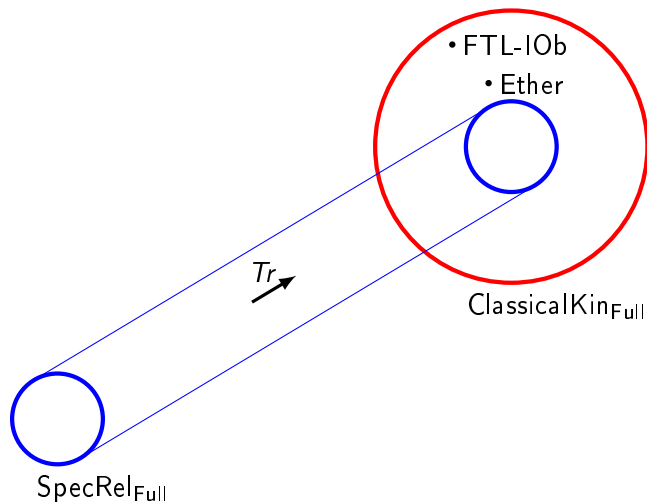
$$\mathbf{Kin} := \{ \text{AxEField}, \text{AxEv}, \text{AxSelf}, \text{AxSymD}, \text{AxLine}, \text{AxTriv}, \text{AxNoAcc} \}$$
$$\mathbf{ClassicalKin}_{Full} := \mathbf{Kin} \cup \{ \text{AxEther}, \text{AbsTime}, \text{AxThExp}_+ \}$$
$$\mathbf{SpecRel}_{Full} := \mathbf{Kin} \cup \{ \text{AxPh}_c, \text{AxThExp} \}$$

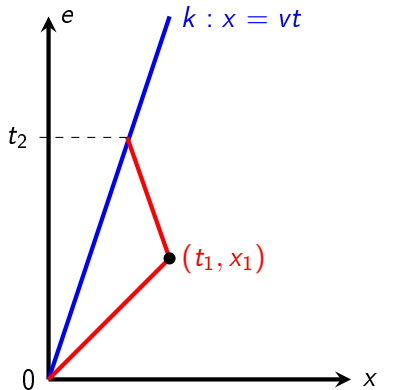
$\mathbf{ClassicalKin}_{Full} \vdash$ Worldview transformations are Galilean transformations.

$\mathbf{SpecRel}_{Full} \vdash$ Worldview transformations are Poincaré transformations (see “Big Book” by H. Andréka, J. X. Madarász, and I. Németi).

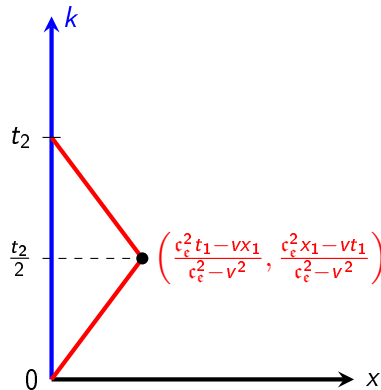
Theorem:

There is an interpretation Tr of SpecRel_{Full} in $\text{ClassicalKin}_{Full}$.

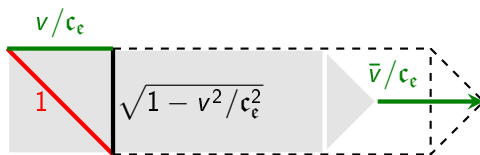




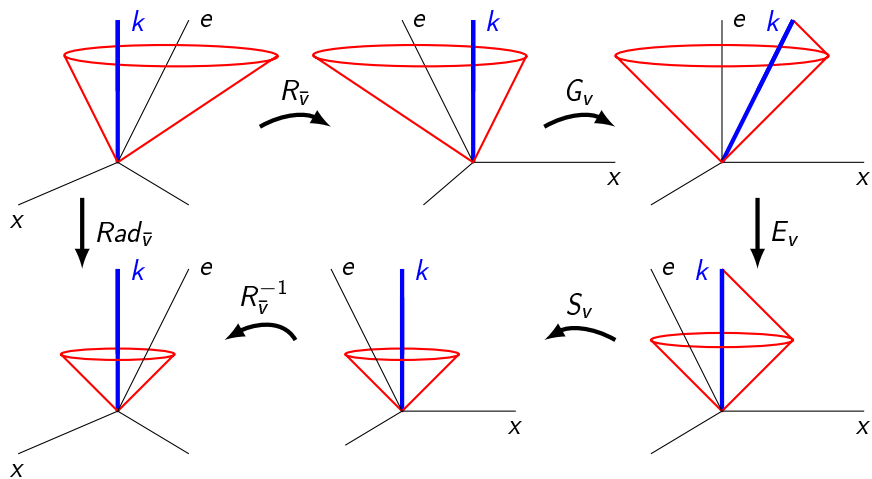
$$E_v := \begin{bmatrix} \frac{1}{1-v^2/c_e^2} & \frac{-v/c_e^2}{1-v^2/c_e^2} \\ \frac{-v}{1-v^2/c_e^2} & \frac{1}{1-v^2/c_e^2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{\sqrt{1-v^2/c_e^2}} & 0 \\ 0 & \frac{1}{\sqrt{1-v^2/c_e^2}} \end{bmatrix}$$



$$S_v := \sqrt{1 - v^2/c_e^2}$$



$$Rad_{\bar{v}} = R_{\bar{v}}^{-1} \circ S_v \circ E_v \circ G_v \circ R_{\bar{v}}$$

$$\text{Tr}(a + b = c) \stackrel{\text{def}}{\equiv} a + b = c$$

$$\text{Tr}(a \cdot b = c) \stackrel{\text{def}}{\equiv} a \cdot b = c$$

$$\text{Tr}(a < b) \stackrel{\text{def}}{\equiv} a < b$$

$$\text{Tr}(\text{W}^{\text{SR}}(k, b, t, x, y, z)) \stackrel{\text{def}}{\equiv}$$

$$\exists t'x'y'z'[\text{W}^{\text{CK}}(k, b, t', x', y', z') \wedge \text{Rad}_k(t', x', y', z') = (t, x, y, z)]$$

$$\text{Tr}(\text{IOb}^{\text{SR}}(k)) \stackrel{\text{def}}{\equiv} \text{IOb}^{\text{CK}}(k) \wedge \forall e[\text{Ether}(e) \rightarrow \text{speed}_e^{\text{CK}}(k) < c_e]$$

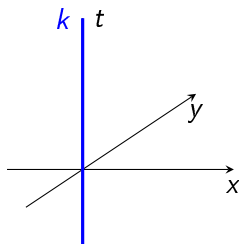
where

$$\text{Ether}(e) \stackrel{\text{def}}{\iff} \text{IOb}^{\text{CK}}(e) \wedge \forall p[\text{Ph}(p) \rightarrow \text{speed}_e^{\text{CK}}(p) = c_e]$$

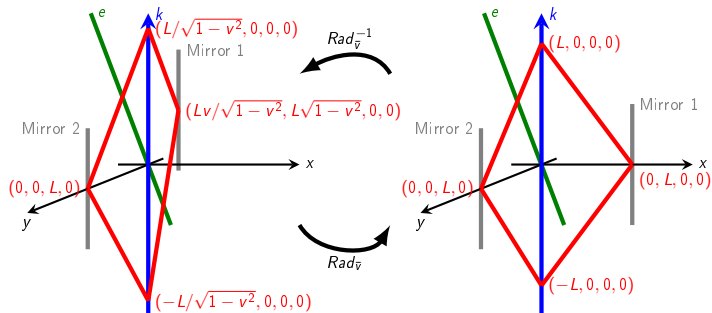
Example: Translation of $AxSelf^{SR}$

$$Tr[\forall k \text{ IOb}^{SR}(k) \rightarrow \forall \bar{x} (\text{W}^{SR}(k, k, \bar{x}) \leftrightarrow x_2 = x_3 = x_4 = 0)]$$

$$\begin{aligned} &\equiv \forall k [\text{IOb}^{CK}(k) \wedge \forall e [\text{Ether}(e) \rightarrow \text{speed}_e^{CK}(k) < c_c]] \\ &\rightarrow \forall \bar{x} (\exists \bar{y} [\text{W}^{CK}(k, k, \bar{y}) \wedge \text{Rad}_k(\bar{y}) = \bar{x}] \leftrightarrow x_2 = x_3 = x_4 = 0)] \end{aligned}$$



Example: the Michelson–Morley Experiment



Note that while our translation function Tr translates axioms of special relativity theory into theorems of classical kinematics, models are transformed the other way round from classical mechanics to special relativity theory.

Theorem:

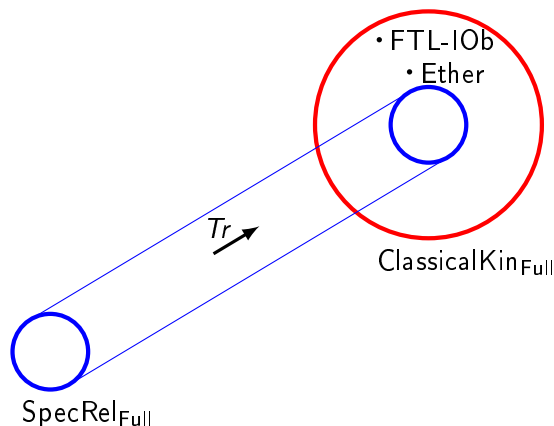
There is an interpretation Tr of $\mathbf{SpecRel}_{Full}$ in $\mathbf{ClassicalKin}_{Full}$.

$$\mathbf{ClassicalKin}_{Full} \vdash Tr(\mathbf{SpecRel}_{Full})$$

- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxEField}^{SR})$
- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxEv}^{SR})$
- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxSelf}^{SR})$
- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxSymD}^{SR})$
- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxLine}^{SR})$
- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxTriv}^{SR})$
- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxNoAcc}^{SR})$
- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxPh}_c)$
- $\mathbf{ClassicalKin}_{Full} \vdash Tr(\text{AxThExp})$

Theorem:

There is no definitional equivalence between SpecRel_{Full} and $\text{ClassicalKin}_{Full}$.



$\text{Conceptual_distance}(\text{SpecRel}_{Full}, \text{ClassicalKin}_{Full}) > 0$

ClassicalKin^{STL}_{Full} :=
 Kin \cup {AxEther, AbsTime, AxThExp^{STL}, AxNoFTL}

AxNoFTL :

All *inertial observers* move *slower than light* with respect to the ether frames.

$$\neg \exists m (IOb(m) \wedge \exists e [Ether(e) \wedge Speed_e^{CK}(m) \geq c_e]).$$

AxThExp^{STL} :

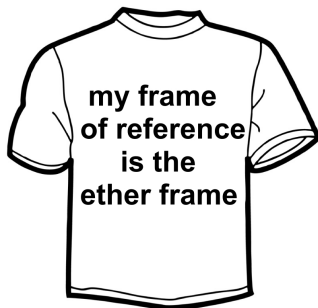
Inertial observers can move with any *speed* which is in the ether frame *slower than that of light*.

$$\begin{aligned} \exists h (IOb(h)) \wedge \forall e \bar{x} \bar{y} (Ether(e) \wedge \text{space}(\bar{x}, \bar{y}) < c_e \cdot \text{time}(\bar{x}, \bar{y})) \\ \rightarrow \exists k IOb(k) \wedge W(e, k, \bar{x}) \wedge W(e, k, \bar{y})). \end{aligned}$$

$$\text{SpecRel}_{Full}^e := \text{SpecRel}_{Full} \cup \{\text{AxPrimitiveEther}\}$$

AxPrimitiveEther :

There is a non-empty class of ether observers, stationary with respect to each other, which is closed under trivial transformations.



$$\exists e (E(e) \wedge \forall k [[IOb(k) \wedge (\exists T \in Triv) w_{ek}^{SR} = T] \leftrightarrow E(k)]])$$

Theorem:

There is an interpretation Tr'_+ of **ClassicalKin^{STL}_{Full}** in **SpecRel^e_{Full}**.

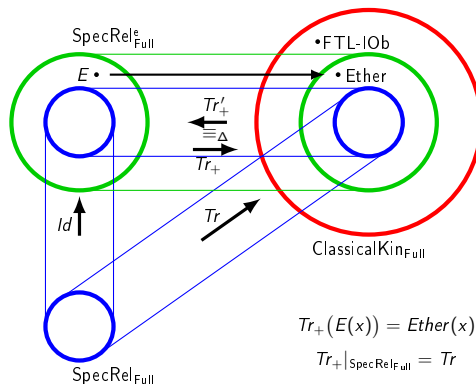
$$\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{ClassicalKin}_{Full}^{STL})$$

- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxEField}^{CK})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxEv}^{CK})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxSelf}^{CK})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxSymD}^{CK})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxLine}^{CK})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxTriv}^{CK})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxNoAcc}^{CK})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxEther})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxAbsTime})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxThExp}^{STL})$
- $\text{SpecRel}_{Full}^e \vdash Tr'_+(\text{AxNoFTL})$

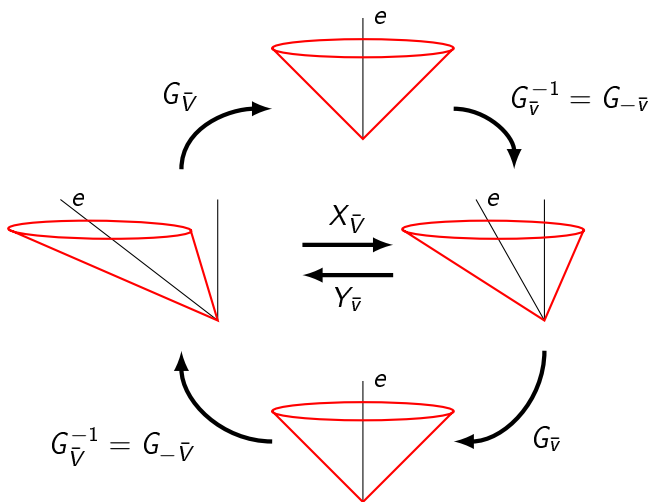
Theorem:

Tr_+ and Tr'_+ are a definitional equivalence between SpecRel_{Full}^e and $\text{ClassicalKin}_{Full}^{STL}$.

$$Tr'_+ \circ Tr_+(\text{SpecRel}_{Full}^e) \Leftrightarrow \text{SpecRel}_{Full}^e \text{ and } Tr_+ \circ Tr'_+(\text{ClassicalKin}_{Full}^{STL}) \Leftrightarrow \text{ClassicalKin}_{Full}^{STL}$$



$$\text{Conceptual_distance}(\text{SpecRel}_{Full}^e, \text{ClassicalKin}_{Full}^{STL}) \leq 2$$



$$\bar{v} = \frac{c_e \bar{V}}{1 + |\bar{V}|} \quad \bar{V} = \frac{\bar{v}}{c_e - |\bar{v}|}$$

Theorem:

There is an interpretation Tr_* of **ClassicalKin_{Full}^{STL}** in **ClassicalKin_{Full}**.

$$\text{ClassicalKin}_{Full} \vdash Tr_*(\text{ClassicalKin}_{Full}^{STL})$$

- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxEField}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxEv}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxSelf}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxSymD}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxLine}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxTriv}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxNoAcc}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxEther}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxAbsTime}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxThExp}^{STL})$
- $\text{ClassicalKin}_{Full} \vdash Tr_*(\text{AxNoFTL})$

Theorem:

There is an interpretation Tr'_* of **ClassicalKin_{Full}** in **ClassicalKin_{Full}^{STL}**.

$$\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(\mathbf{ClassicalKin}_{Full})$$

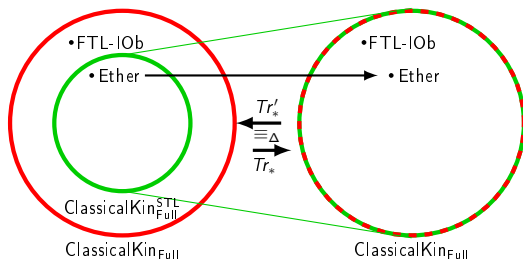
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxEField^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxEv^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxSelf^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxSymD^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxLine^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxTriv^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxNoAcc^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxEther^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxAbsTime^{CK})$
- $\mathbf{ClassicalKin}_{Full}^{STL} \vdash Tr'_*(AxThExp_+)$

Theorem:

Tr_* and Tr'_* are a definitional equivalence between $\text{ClassicalKin}_{Full}^{STL}$ and $\text{ClassicalKin}_{Full}$.

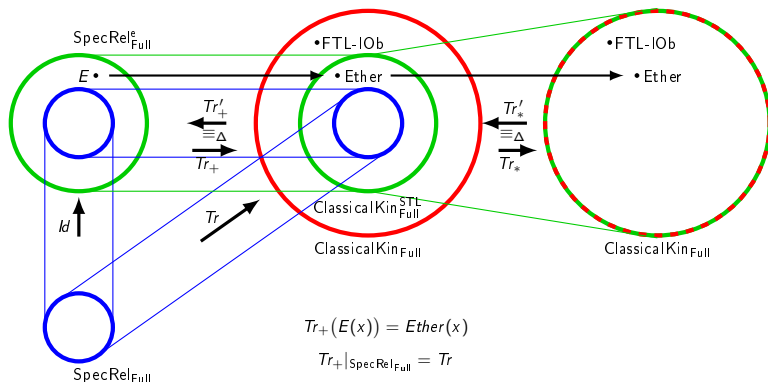
$Tr'_* \circ Tr_*(\text{ClassicalKin}_{Full}^{STL}) \Leftrightarrow \text{ClassicalKin}_{Full}^{STL}$ and

$Tr_* \circ Tr'_*(\text{ClassicalKin}_{Full}) \Leftrightarrow \text{ClassicalKin}_{Full}$



Theorem:

$Tr_* \circ Tr_+$ and $Tr'_+ \circ Tr'_*$ are a definitional equivalence between $SpecRel_{Full}^e$ and $ClassicalKin_{Full}$.



$Conceptual_distance(SpecRel_{Full}, ClassicalKin_{Full}) = 1$

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Koen Lefever

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Koen Lefever and Gergely Székely

“Comparing Classical and Relativistic Kinematics in First-Order Logic”

Forthcoming in *Logique et Analyse*, 2018

Pre-print: <http://arxiv.org/abs/1707.05371>

or <http://philsci-archive.pitt.edu/13200>